Why Are Proofs Relevant in Proof-Relevant Models?

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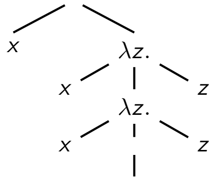
Approximation Theory

λ-calculus

terms : $\Lambda : M, N \ ::= \ x \mid \lambda x.M \mid (MN)$

β-reduction : $(\lambda x.M)N \mapsto_\beta M\{N/x\}$

Böhm Trees



$\mathcal{BT}(Yx)$ avec $Yx =_\beta x(\lambda z.Yxz)$

Denotationnal Models

- intersection type assignment system

$$\alpha ::= a \mid \alpha_1 \wedge \cdots \wedge \alpha_n \to \alpha$$

- Interpretation of $M \in \Lambda$

$$[\![M]\!] = \{(\Gamma\alpha) \mid \Gamma \vdash M : \alpha\}$$

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λ-calculus

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terms : $\Lambda : M, N ::= x \mid \lambda x.M \mid (MN)$

β-reduction : $(\lambda x.M)N \mapsto_\beta M\{N/x\}$

M is in

- head normal form if $M = \lambda x_1 \dots x_n.x_j M_1 \cdots M_k$.
- normal form if $M = \lambda x_1 \dots x_n.x_j M_1 \cdots M_k$ and the M_is are in normal forms.

Böhm Tree

$\Lambda_\perp$: $\quad L, O ::= \perp \mid x \mid \lambda x.M \mid MN$

$\leq_\perp$ least compatible preorder s.t. $\forall L \in \Lambda_\perp, \perp \leq L$

$\mathcal{A}$: $\quad A, B ::= \perp \mid \lambda x_1 \dots x_n.yA_1 \cdots A_k \quad (\text{for } n, k \geq 0)$

$$\mathcal{A}(M) = \{A \in \mathcal{A} \mid \exists N \in \Lambda \,.\, M \twoheadrightarrow_\beta N \text{ and } A \leq_\perp N\}$$

$$\mathrm{BT}(M) = \bigvee \mathcal{A}(M)$$

Examples

$$
\begin{array}{lll}
I = \lambda x.x & \mathcal{A}(\mathsf{I}) = \{\bot, \lambda x.x\} & \mathrm{BT}(\mathsf{I}) = \lambda x.x \\
1 = \lambda xy.xy & \mathcal{A}(1) = \{\bot, \lambda xy.x\bot, \lambda xy.xy\} & \mathrm{BT}(1) = \lambda xy.xy \\
\Delta = \lambda x.xx & \mathcal{A}(\Delta) = \{\bot, \lambda x.x\bot, \lambda x.xx\} & \mathrm{BT}(\Delta) = \lambda x.xx \\
\Omega = \Delta\Delta & \mathcal{A}(\Omega) = \{\bot\} & \mathrm{BT}(\Omega) = \bot
\end{array}
$$

$$\mathsf{Y} = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$$

$$\mathcal{A}(\mathsf{Y}) = \{\bot\} \cup \{\lambda f.f^n(\bot) \mid n > 0\}$$

$$\mathrm{BT}(\mathsf{Y}) = \lambda f.f(f(f(f(f(\cdots)))))$$

Distributor

A distributor $F : A \nrightarrow B$ between A, B small categories is a functor $F : B^{op} \times A \to \mathit{Set}$.

We have *Dist* a cartesian closed category of distributors.
With cartesian product $A \otimes B = A \times B$ and exponential objects $A \Rightarrow B = A^{op} \times B$.

Symmetric strict monoidal completion

Let A be a small category. The symmetric strict monoidal completion $!A$ of A is the category:

- $!A = \{\langle a_1, \ldots, a_n \rangle \mid a_i \in A \text{ and } n \in \mathbb{N}\}$;
- $!A[\langle a_1, \ldots, a_n \rangle, \langle a'_1, \ldots, a'_{n'} \rangle] = \begin{cases} \{\langle \sigma, f_i \rangle_{i \in [n]} \mid f_i : a_i \to a'_{\sigma(i)},\ \sigma \in S_n\}, \text{if } n = n'; \\ \emptyset, \text{otherwise;} \end{cases}$
- for $f = \langle \sigma, f_i \rangle_{i \in [n]} : \vec{a} \to \vec{b}$ and $g = \langle \tau, g_i \rangle_{i \in [n]} : \vec{b} \to \vec{c}$ their composition is defined as follows $g \circ f = \langle \tau\sigma, g_{\sigma(1)} \circ f_1, \ldots, g_{\sigma(n)} \circ f_n \rangle$;
- for $\vec{a} = \langle a_1, \ldots, a_n \rangle \in !A$, the identity on $\vec{a}$ is given by $1_{\vec{a}} = \langle 1_n, 1_{a_1}, \ldots, 1_{a_n} \rangle$;
- the monoidal structure $\vec{a} \oplus \vec{b}$ is given by list concatenation.

CatSym

The endofunctor $! : \mathrm{Cat} \to \mathrm{Cat}$, can be lifted to a pseudocomonad over Dist, we denote as CatSym its Klesli bicategory:

- $Ob(\mathrm{CatSym})$ are the small categories
- For $A, B \in \mathrm{CatSym}$, we have $\mathrm{CatSym}(A, B) = Dist(B, !A)$.
- The identity $1_A(\vec{a}, a) = !A(\vec{a}, \langle a \rangle)$.
- For $F : A \nrightarrow B$ and $G : B \nrightarrow C$, composition is given by $(G \circ F)(a, c) = \int^{b \in !B} G(b, c) \times F(a, b)$.
- CatSym is cartesian, with cartesian product the disjoint union $A \& B = A \bigsqcup B$. The terminal object is the empty category.
- CatSym is cartesian closed, with exponential object $A \Rightarrow B = !A^{\mathrm{op}} \times B$.

Pseudo-reflexive Object

Given a small category A, we define an inductive family of small categories:

$$D_0 = A, \qquad D_{n+1} = (!D_n^{op} \times D_n) \sqcup A.$$

Then, we construct a family of functors $\iota_n : D_n \hookrightarrow D_{n+1}$, again by induction:

$$\iota_0 = \mathsf{in}_A, \qquad \iota_{n+1} = (!(\iota_n)^{op} \times \iota_n) \sqcup 1_A.$$

Directed colimit $D_A = lim_{n \in \mathbb{N}} D_n$
Free algebra $\langle D_A, \iota : !D_A^{op} \times D_A \to D_A \rangle$ with a retraction pair
$D_A \Rightarrow D_A \lhd D_A$

Interpretation

A bicategorical model $\mathcal{D} = \langle D, \alpha, i, j \rangle$ in C, where $\langle i, j \rangle$ the retraction pair and $\alpha : id_{D \Rightarrow D} \cong j \circ i$.

Interpretation

The interpretation of a λ-term M (with $\mathrm{FV}(M) \subseteq \vec{x}$) in $\mathcal{D}$ is a 1-cell $[\![M]\!]_{\vec{x}} : D^{\& n} \to D \qquad (= (D \& \ldots \& D) \to D)$

$$
\begin{array}{lcl}
[\![x_i]\!]_{\vec{x}} & = & \pi_i^n, \\
[\![\lambda y.M]\!]_{\vec{x}} & = & i \circ \lambda\big([\![M]\!]_{\vec{x},y}\big), \ \textit{wlog assume } y \notin \vec{x}, \\
[\![MN]\!]_{\vec{x}} & = & ev_{D,D} \circ \langle j \circ [\![M]\!]_{\vec{x}}, [\![N]\!]_{\vec{x}} \rangle.
\end{array}
$$

Theorems

Lemma of Substitution: If $M \in \Lambda^o(\vec{x}, y), N \in \Lambda^o(\vec{x})$ and $y \notin \vec{x}$ then $[\![M[N/y]]\!]_{\vec{x}} \cong [\![M]\!]_{\vec{x},y} \circ \langle 1_{D^{\&\mathsf{len}(\vec{x})}}, [\![N]\!]_{\vec{x}} \rangle$

Theorem of Soundness

$M, N \in \Lambda^o(\vec{x})$, if $M \to_\beta N$ then $[\![M \to_\beta N]\!]_{\vec{x}} : [\![M]\!]_{\vec{x}} \cong [\![N]\!]_{\vec{x}}$

Theorem : semantic sound with respect to confluence

If $M \twoheadrightarrow_\beta L \twoheadrightarrow_\beta N$ and $M \twoheadrightarrow_\beta L' \twoheadrightarrow_\beta N$. Then

$$[\![L \twoheadrightarrow_\beta N]\!]_{\vec{x}} \star [\![M \twoheadrightarrow_\beta L]\!]_{\vec{x}} = [\![L' \twoheadrightarrow_\beta N]\!]_{\vec{x}} \star [\![M \twoheadrightarrow_\beta L']\!]_{\vec{x}}$$

System R

$$\text{Types:} \quad a, b, c := o \mid \langle a_1, \ldots, a_k \rangle \Rightarrow a$$

$$\frac{f \in A(o, o')}{f : o \to o'} \qquad \frac{\langle \sigma, \vec{f} \rangle : \vec{a}' \to \vec{a} \qquad f : a \to a'}{\langle \sigma, \vec{f} \rangle \Rightarrow f : (\vec{a} \Rightarrow a) \to (\vec{a}' \Rightarrow a')}$$

$$\frac{\sigma \in \mathfrak{S}_k \quad f_1 : a_1 \to a'_{\sigma(1)} \quad \ldots \quad f_k : a_k \to a'_{\sigma(k)}}{\langle \sigma, f_1, \ldots, f_k \rangle : \langle a_1, \ldots, a_{k'} \rangle \to \langle a'_1, \ldots, a'_k \rangle}$$

Figure: Category of Intersection Types D

$$\frac{f : a' \to a}{x_1 : \langle\rangle, \ldots, x_i : \langle a' \rangle, \ldots, x_n : \langle\rangle \vdash x_i : a}\ \text{ax} \qquad \frac{\Delta, x : \vec{a} \vdash M : a}{\Delta \vdash \lambda x.M : \vec{a} \Rightarrow a}\ \text{abs}$$

$$\frac{\Gamma_0 \vdash M : \langle a_1, \ldots, a_k \rangle \Rightarrow a \qquad (\Gamma_i \vdash N : a_i)_{i=1}^k \qquad \eta : \Delta \to \bigotimes_{i=0}^k \Gamma_i}{\Delta \vdash MN : a}\ \text{app}$$

Figure: Derivations and Typing of System R.

Congruence

$$\frac{\begin{matrix}\pi_0\\ \vdots\\ \Gamma_0 \vdash \vec{b} \Rightarrow a\end{matrix} \qquad \left(\begin{matrix}[f_i]\pi_{\sigma^{-1}(i)}\\ \vdots\\ \Gamma_{\sigma^{-1}(i)} \vdash b_i\end{matrix}\right)_{i=1}^{k} \qquad (1 \otimes (\sigma^{-1})^{\star}) \circ \eta}{\Delta \vdash a} \sim \frac{\begin{matrix}[\langle \sigma, \vec{f}\rangle \Rightarrow 1]\pi_0\\ \vdots\\ \Gamma_0 \vdash \vec{a} \Rightarrow a\end{matrix} \qquad \left(\begin{matrix}\pi_i\\ \vdots\\ \Gamma_i \vdash a_i\end{matrix}\right)_{i=1}^{k} \qquad \eta}{\Delta \vdash a}$$

$$\frac{\begin{matrix}\pi_0\{\theta_0\}\\ \vdots\\ \Gamma_0 \vdash \vec{a} \Rightarrow a\end{matrix} \qquad \left(\begin{matrix}\pi_i\{\theta_i\}\\ \vdots\\ \Gamma_i \vdash a_i\end{matrix}\right)_{i=1}^{k} \qquad \eta : \Delta \to \otimes_{j=0}^{k} \Gamma_j}{\Delta \vdash a} \sim \frac{\begin{matrix}\pi_0\\ \vdots\\ \Gamma'_0 \vdash \vec{a} \Rightarrow a\end{matrix} \qquad \left(\begin{matrix}\pi_i\\ \vdots\\ \Gamma'_i \vdash a_i\end{matrix}\right)_{i=1}^{k} \qquad (\otimes_{j=0}^{k} \theta_j) \circ \eta}{\Delta \vdash a}$$

$$\frac{\begin{matrix}[g]\pi\{1 \oplus \langle \sigma, \vec{g}\rangle\}\\ \vdots\\ \Delta, \vec{a}' \vdash a'\end{matrix} \qquad f : (\vec{a} \Rightarrow a) \to b}{\Delta \vdash b} \sim \frac{\begin{matrix}\pi\\ \vdots\\ \Delta, \vec{a} \vdash a\end{matrix} \qquad f \circ (\langle \sigma, \vec{g}\rangle \Rightarrow g) : (\vec{a}' \Rightarrow a') \to b}{\Delta \vdash b}$$

where $\langle \sigma, f_1, \ldots, f_k\rangle : \vec{a} = \langle a_1, \ldots, a_k\rangle \to \vec{b} = \langle b_1, \ldots, b_k\rangle$, $\langle \sigma, \vec{g}\rangle : \vec{a}' \to \vec{a}$, $g : a \to a'$ and $\theta_i : \Gamma_i \to \Gamma'_i$.

Figure: Congruence on derivations.

Example

Let $k \in \mathbb{N}$, $\sigma \in \mathfrak{S}_k$ and $\pi =$

$$\dfrac{\begin{array}{c}\pi_0 \\ \vdots \\ \Gamma_0 \vdash \langle a_1, \ldots, a_k \rangle \Rightarrow a\end{array} \qquad \left(\begin{array}{c}\pi_i \\ \vdots \\ \Gamma_i \vdash a_i\end{array}\right)_{i=1}^{k} \qquad \eta}{\Delta \vdash a}$$

Let $\eta' = (1 \otimes (\sigma)^\star) \circ \eta$ and $\pi' =$

$$\dfrac{\begin{array}{c}\pi_0[\sigma \Rightarrow a] \\ \vdots \\ \Gamma_0 \vdash \langle a_{\sigma(1)}, \ldots, a_{\sigma(k)} \rangle \Rightarrow a\end{array} \qquad \left(\begin{array}{c}\pi_{\sigma(i)} \\ \vdots \\ \Gamma_{\sigma(i)} \vdash a_{\sigma(i)}\end{array}\right)_{i=1}^{k} \qquad \eta'}{\Delta \vdash a}$$

$\pi \sim \pi'$ by the first rule of congruence.

Left action on derivations

$$[g : a \to b]\left(\frac{f : a' \to a}{\langle\rangle, \ldots, \langle a' \rangle, \ldots, \langle\rangle \vdash a}\right) = \frac{g \circ f : a' \to b}{\langle\rangle, \ldots, \langle a' \rangle, \ldots, \langle\rangle \vdash b}$$

$$[g : a' \to b]\left(\frac{\begin{matrix}\pi \\ \vdots \\ \Delta, \vec{a} \vdash a\end{matrix} \qquad f : (\vec{a} \Rightarrow a) \to a'}{\Delta \vdash a}\right) = \frac{\begin{matrix}\pi \\ \vdots \\ \Delta, \vec{a} \vdash a\end{matrix} \qquad g \circ f : (\vec{a} \Rightarrow a) \to b}{\Delta \vdash b}$$

$$[g : a \longrightarrow b]\left(\frac{\begin{matrix}\pi_0 \\ \vdots \\ \Gamma_0 \vdash \vec{a} \Rightarrow a\end{matrix} \qquad \left(\begin{matrix}\pi_i \\ \vdots \\ \Gamma_i \vdash a_i\end{matrix}\right)_{i=1}^{k} \qquad \eta : \Delta \longrightarrow \bigotimes_0^k \Gamma_j}{\Delta \vdash a}\right) = \frac{\begin{matrix}[1 \Rightarrow g]\pi_0 \\ \vdots \\ \Gamma_0 \vdash \vec{a} \Rightarrow b\end{matrix} \qquad \left(\begin{matrix}\pi_i \\ \vdots \\ \Gamma_i \vdash a_i\end{matrix}\right)_{i=1}^{k} \qquad \eta}{\Delta \vdash b}$$

where $\vec{a} = \langle a_1, \ldots, a_k \rangle$.

Type Distributor

Type Distributor $\mathsf{T}^{D}_{\vec{x}}(M) : !D^n \nrightarrow D$,

1. objects

$$\mathsf{T}^{D}_{\vec{x}}(M)(\Delta, a) = \{\tilde{\pi} \in R_{\rightarrow}/\sim \mid \pi \triangleright \Delta \vdash M : a\}$$

2. morphisms

$$\mathsf{T}^{D}_{\vec{x}}(M)(f, \eta) : \mathsf{T}^{D}_{\vec{x}}(M)(\Delta, a) \rightarrow \mathsf{T}^{D}_{\vec{x}}(M)(\Delta', a')$$

$$\tilde{\pi} \mapsto \widetilde{[f]\pi\{\eta\}}$$

Equivalence

Theorem

For all $M \in \Lambda_\perp$, there is a natural isomorphism

$$\mathsf{itd}^M_{\vec{x}} : \mathsf{T}^D_{\vec{x}}(M) \cong [\![M]\!]_{\vec{x}} \circ_{\mathrm{Dist}} \overline{\mu}_1.$$

$\mu_1 : !A_1 \times \cdots \times !A_n \to !(A_1 \sqcup \cdots \sqcup A_n)$ and $\overline{\mu}_1$ the corresponding distributor

when $M \to_\beta N$ we also get an isomorphism:

$$\mathsf{T}^D_{\vec{x}}(M \to_\beta N) : \mathsf{T}^D_{\vec{x}}(M) \cong \mathsf{T}^D_{\vec{x}}(N)$$

.

Extend Notions to Böhm Trees

$\mathsf{T}^{D}_{\vec{x}}(\bot) = \emptyset_{!D^n,D}$

Lemma: If $L \leq_\bot P$ then $[\![L]\!]_{\vec{x}} \subseteq [\![P]\!]_{\vec{x}}$ and $\mathsf{T}^{D}_{\vec{x}}(L) \subseteq \mathsf{T}^{D}_{\vec{x}}(P)$.

Consider $\langle \mathcal{A}(M), \leq_\bot \rangle$,

$$\begin{aligned} [\![-]\!]_{\vec{x}} : \mathcal{A}(M) &\to \mathrm{Dist}(!(D^{\&n}), D) \\ A &\mapsto [\![A]\!]_{\vec{x}}, \\ A \leq_\bot A' &\mapsto [\![A]\!]_{\vec{x}} \subseteq [\![A']\!]_{\vec{x}}. \end{aligned}$$

$[\![\mathrm{BT}(M)]\!]_{\vec{x}} = lim_{A \in \mathcal{A}(M)} [\![A]\!]_{\vec{x}}$.

$\mathsf{T}^{D}_{\vec{x}}(\mathrm{BT}(M)) : !D^n \nrightarrow D$

$\mathsf{T}^{D}_{\vec{x}}(\mathrm{BT}(M))(\Delta, a) = \bigcup_{A \in \mathcal{A}(M)} \mathsf{T}^{D}_{\vec{x}}(A)(\Delta, a)$

$\mathsf{T}^{D}_{\vec{x}}(\mathrm{BT}(M))(\eta, f)(\tilde{\pi}) = \widetilde{[f]\pi\{\eta\}}$
with $\eta : \Delta' \to \Delta, f : a \to a'$

Theorem

A natural isomorphism $[\![\mathrm{BT}(M)]\!]_{\vec{x}} \circ_{\mathrm{Dist}} \overline{\mu}_1 \cong \mathsf{T}^{D}_{\vec{x}}(\mathrm{BT}(M))$.

Some Definitions

- Measure $\mathsf{s}(\pi) = n$ if π contains exactly n times the rule (app).
- Given a derivation $\pi \in R_{\rightarrow}$, a β-redex of π is a subderivation of π having shape:

$$\dfrac{\dfrac{\Gamma_0, \langle a_1, \ldots, a_k \rangle \vdash a}{\Gamma_0 \vdash \langle a_1, \ldots, a_k \rangle \Rightarrow a} \qquad \begin{array}{c}\vdots \\ (\Gamma_i \vdash a_i)_{i=1}^{k}\end{array} \qquad \eta : \Delta \to \bigotimes_{i=0}^{k} \Gamma_i}{\Delta \vdash a}$$

- Assume that $\pi \triangleright \Delta \vdash M : a$. A redex R of M is informative in π if it is typed by a redex of π.
- A derivation π is in β-normal form if it has no β-redexes as subderivations.

Some Definitions

The set of subterm occurrences of M that are typed in π, depending of the last rule of π:

- (ax) $\pi = \dfrac{f : a' \to a}{x_1 : \langle\rangle, \ldots, x_i : \langle a' \rangle, \ldots, x_n : \langle\rangle \vdash x_i : a}$ $\quad \mathrm{tocc}(\pi) = \{[]\}$;

- (abs) $\pi = \dfrac{\Delta, x : \vec{a} \vdash M' : a}{\Delta \vdash M = \lambda x.M' : \vec{a} \Rightarrow a}$ where π' is the premise

  $\mathrm{tocc}(\pi) = \{[]\} \cup \{\lambda x.C[] \mid C[] \in \mathrm{tocc}(\pi')\}$;

- (app) where π_0 is the premise corresponding to M_0, and $\pi_1, \ldots, \pi_k$ corresponding to M_1

  $$\pi = \dfrac{\Gamma_0 \vdash M_0 : \langle a_1, \ldots, a_k \rangle \Rightarrow a \qquad (\Gamma_i \vdash M_1 : a_i)_{i=1}^{k} \qquad \eta : \Delta \to \bigotimes_{i=0}^{k} \Gamma_i}{\Delta \vdash M_0 M_1 : a}$$

  $$\begin{aligned}\mathrm{tocc}(\pi) = \{[]\} &\cup \{C[\,]M_1 \mid C[] \in \mathrm{tocc}(\pi_0)\} \\ &\cup \{M_0(C[\,]) \mid C[] \in \textstyle\bigcup_{i=1}^{k} \mathrm{tocc}(\pi_i)\}.\end{aligned}$$

Example

$$\pi = \cfrac{\cfrac{f : a \to a'}{x : \langle\langle\rangle \Rightarrow a\rangle \vdash x : \langle\rangle \Rightarrow a'}}{x : \langle\langle\rangle \Rightarrow a\rangle \vdash x(\mathsf{II})}$$

$$\mathrm{tocc}(\pi) = \{[], [](\mathsf{II})\}$$

Normal Form of Proofs

$\tilde{\pi} \in \mathsf{T}_{\vec{x}}^{D}(M)(\Delta, a)$ for some $\langle \Delta, a \rangle \in !D^{\mathsf{len}(\vec{x})} \times D$.
We say that $\tilde{\pi}$ is normalizable along M if $\exists N \in \Lambda, M \twoheadrightarrow_\beta N$ and $\mathsf{T}_{\vec{x}}^{D}(M \twoheadrightarrow_\beta N)_{\Delta,a}(\tilde{\pi})$ is in normal form.

$$\mathsf{nf}(\mathsf{T}_{\vec{x}}^{D}(M)(\Delta, a)) = \quad \{\tilde{\pi} \in \mathsf{nf}(R_{\to}) \mid \exists N \in \Lambda \,.\, M \twoheadrightarrow_\beta N \text{ and } \tilde{\pi} \in [\![N]\!]_{\vec{x}}(\Delta, a)\}$$

naturally extends to a distributor $\mathsf{nf}(\mathsf{T}_{\vec{x}}^{D}(M))$.

Proposition:
Let $M, N \in \Lambda^o(\vec{x})$ and $\tilde{\pi} \in \mathsf{T}^D_{\vec{x}}(M)(\Delta, a)$. Assume that $M \to_\beta N$ because a redex occurrence R in M is contracted.

1. If R is typed in π then $\mathsf{s}\left(\mathsf{T}^D_{\vec{x}}(M \to_\beta N)_{\Delta,a}(\tilde{\pi})\right) < \mathsf{s}\left(\tilde{\pi}\right)$,
2. Otherwise, we have $\mathsf{T}^D_{\vec{x}}(M \to_\beta N)_{\Delta,a}(\tilde{\pi}) = \tilde{\pi}$.

Theorem

The reduction strategy contracting typed redexes in type derivations along M is strongly normalizing.

For $\tilde{\pi} \in |\mathsf{T}^{D}_{\vec{x}}(M)|$ we denote its normal form $\mathrm{nf}(\tilde{\pi})_M$.

We obtain $\mathrm{nf}(\mathsf{T}^{D}_{\vec{x}}(M)(\Delta, a)) = \{\mathrm{nf}(\tilde{\pi}) \in R_{\rightarrow} \mid \tilde{\pi} \in [\![M]\!]_{\vec{x}}(\Delta, a)\}$.

Theorem

For $M \in \Lambda^{o}(\vec{x})$, there is a canonical natural isomorphism

$$\mathrm{Norm}_{\vec{x}}(M) : \mathsf{T}^{D}_{\vec{x}}(M) \cong \mathrm{nf}(\mathsf{T}^{D}_{\vec{x}}(M))$$

given by normalization $\tilde{\pi} \mapsto \mathrm{nf}(\tilde{\pi})$.

Minimal Terms

Define a map $L^{\vec{x}}_{-} : R_{\to} \to \Lambda_{\perp}$ by induction on the structure of π as follows:

- if π is an axiom, then $L^{\vec{x}}_{\pi} = x_i$, where i is the index of the only type appearing in the type environment of π;
- if π is an abstraction, then $L^{\vec{x}}_{\pi} = \lambda y.(L^{\vec{x},y}_{\pi'})$, where $\pi' \in R_{\to}$ is the unique premise of $\pi \in R_{\to}$ and we can assume $y \notin \vec{x}$;
- if π is an application, then $L^{\vec{x}}_{\pi} = L^{\vec{x}}_{\pi_0}(\bigvee_{i=1}^{k} L^{\vec{x}}_{\pi_i})$ where $\pi_0 \in R_{\to}$ and $\pi_1, \ldots, \pi_k \in R_{\to}$, for some $k \in \mathbb{N}$, are the premises of $\pi \in R$.

We can extend the map to congruence.

Examples

Let $\pi = \cfrac{\cfrac{f : a' \to a}{x : \langle\langle\rangle \Rightarrow a'\rangle \vdash x : \langle\rangle \Rightarrow a}}{x : \langle\langle\rangle \Rightarrow a'\rangle \vdash x\Omega : a}$ $\qquad L^x_\pi = x\bot.$

Interesting Results

Proposition: Let $M \in \Lambda^o(\vec{x})$ and $\pi \in R_{\to}(M)$.

- $\pi \in R_{\to}(L_\pi^{\vec{x}})$ and $L_\pi^{\vec{x}} \leq_\perp M$.
- If π is a normal form then $L_\pi^{\vec{x}} \in \mathcal{A}$, whence $L_\pi^{\vec{x}} \in \mathcal{A}(M)$.

Interesting Results

Proposition: Let $M \in \Lambda^o(\vec{x})$ and $\pi \in R_{\rightarrow}(M)$.

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- If π is a normal form then $L^{\vec{x}}_{\pi} \in \mathcal{A}$, whence $L^{\vec{x}}_{\pi} \in \mathcal{A}(M)$.

Commutation Theorem:

$\forall M \in \Lambda^o(\vec{x})$, $\mathsf{nf}(\mathsf{T}^{D}_{\vec{x}}(M)) = \mathsf{T}^{D}_{\vec{x}}(\mathrm{BT}(M))$.

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Commutation Theorem:

$\forall M \in \Lambda^o(\vec{x})$, $\mathsf{nf}(\mathsf{T}_{\vec{x}}^D(M)) = \mathsf{T}_{\vec{x}}^D(\mathrm{BT}(M))$.

Approximation Theorem:

$\forall M \in \Lambda^o(\vec{x})$ there is a natural isomorphism
$\mathsf{appr}_{\vec{x}}(M) : \mathsf{T}_{\vec{x}}^D(M) \cong \mathsf{T}_{\vec{x}}^D(\mathrm{BT}(M))$.

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Proposition: Let $M \in \Lambda^o(\vec{x})$ and $\pi \in R_{\to}(M)$.

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$\mathsf{appr}_{\vec{x}}(M) : \mathsf{T}^D_{\vec{x}}(M) \cong \mathsf{T}^D_{\vec{x}}(\mathrm{BT}(M))$.

Corollary: The model is sensible.

λ-Theories:

Definition of a λ-theory

Any congruence on Λ (i.e. an equivalence relation compatible with abstraction and application) containing the β-conversion.

$$\mathcal{B} = \{(M, N) \mid \mathrm{BT}(M) = \mathrm{BT}(N)\} \subseteq \Lambda^2,$$

Theory of a Bicategorical Model

A natural isomorphism $\alpha : [\![M]\!]_{\vec{x}} \cong [\![N]\!]_{\vec{x}}$ is coherent wrt β-normalization when the induced natural isomorphism $\alpha : \mathsf{T}^{D}_{\vec{x}}(M) \cong \mathsf{T}^{D}_{\vec{x}}(N)$ satisfies: for all $\tilde{\pi} \in \mathsf{T}^{D}_{\vec{x}}(M)(\Delta, a)$ we have $\mathsf{nf}(\tilde{\pi}) = \mathsf{nf}(\alpha_{\Delta,a}(\tilde{\pi}))$.

Theory of $\mathcal{D}$ in CatSym

$$\mathrm{Th}(\mathcal{D}) = \{(M, N) \mid M, N \in \Lambda^o(\vec{x}) \text{ and } \alpha : [\![M]\!]_{\vec{x}} \cong [\![N]\!]_{\vec{x}} \ \alpha \textit{ coherent wrt } \beta\textit{-normalization}\}$$

Characterization

Characterization of the Theory

$\mathsf{T}^{D}_{\vec{x}}(M) \cong \mathsf{T}^{D}_{\vec{x}}(N)$ iff $\mathrm{BT}(M) = \mathrm{BT}(N)$.

$$\text{Corrolary}: \ \mathrm{Th}(\mathcal{D}) = \mathcal{B}.$$

Conclusion

Merci de votre attention

Actions

$$\left(\frac{f : a' \to a}{\langle\rangle, \ldots, \langle a' \rangle, \ldots, \langle\rangle \vdash a}\right)\{g : b \to a'\} \quad = \quad \frac{f \circ g}{\langle\rangle, \ldots, \langle b \rangle, \ldots, \langle\rangle \vdash a}$$

$$\left(\frac{\begin{array}{c}\pi\\ \vdots\\ \Delta, \vec{a} \vdash a\end{array} \qquad f : (\vec{a} \Rightarrow a) \to b}{\Delta \vdash b}\right)\{\eta\} \quad = \quad \frac{\begin{array}{c}\pi\{\eta \oplus \langle 1 \rangle\}\\ \vdots\\ \Delta', \vec{a} \vdash a\end{array} \qquad f : (\vec{a} \Rightarrow a) \to b}{\Delta' \vdash \vec{a} \Rightarrow a}$$

$$\left(\frac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma_0 \vdash \vec{a} \Rightarrow a\end{array} \qquad \left(\begin{array}{c}\pi_i\\ \vdots\\ \Gamma_i \vdash a_i\end{array}\right)_{i=1}^{k} \qquad \theta : \Delta \to \bigotimes_{j=0}^{k} \Gamma_j}{\Delta \vdash a}\right)\{\eta\} = \frac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma_0 \vdash \vec{a} \Rightarrow a\end{array} \qquad \left(\begin{array}{c}\pi_i\\ \vdots\\ \Gamma_i \vdash a_i\end{array}\right)_{i=1}^{k} \qquad \theta \circ \eta}{\Delta' \vdash a}$$

where $\vec{a} = \langle a_1, \ldots, a_k \rangle$ and $\eta : \Delta' \to \Delta$.

Figure: Right action on derivations.

Bicategory C

- objects $A, B \in Ob(\mathsf{C})$ also called 0-cells;
- for all $A, B \in \mathsf{C}$, a category $\mathsf{C}(A, B)$;
  objects in $\mathsf{C}(A, B)$ named 1-cells or morphisms from A to B;
  arrows in $\mathsf{C}(A, B)$ (between 1-cells) named 2-cells;
  composition of 2-cells called vertical composition;
- for every $A, B, C \in \mathsf{C}$, a bifunctor called horizontal composition $\circ_{A,B,C} \colon \mathsf{C}(A, C) \times \mathsf{C}(A, B) \to \mathsf{C}(A, C)$;
- for every $A \in \mathsf{C}$, a functor $1_A \colon 1 \to \mathsf{C}(A, A)$;
- for all 1-cells $F \colon A \to B$, $G \colon B \to C$, and $H \colon C \to D$, a family of invertible 2-cells expressing the associativity law
  $\alpha_{H,G,F} \colon H \circ (G \circ F) \cong (H \circ G) \circ F$;
- for every 1-cell $F \colon A \to B$, two families of invertible 2-cells expressing the identity law
  $\lambda_F \colon 1_B \circ F \cong F, \quad \rho_F \colon F \cong F \circ 1_A$.

Bicategory of Distributors

- 0-cells are small categories $A, B, C, \ldots$
- 1 cells $F : A \nrightarrow B$ are functors $F : A^{\mathrm{op}} \times B \to \mathrm{Set}$.
- 2-cells $\alpha : F \Rightarrow G$ are natural transformations.
- For fixed 0-cells A and B, the 1-cells and 2-cells are organized as a category $\mathrm{Dist}(A, B)$.
- For $A \in \mathrm{Dist}$, the identity $1_A : A \nrightarrow A$ is defined as the Yoneda embedding $1_A(a, a') = A(a, a')$.
- For 1-cells $F : A \nrightarrow B$ and $G : B \nrightarrow C$, the *horizontal composition* is given by

  $$(G \circ F)(a, c) = \int^{b \in B} G(b, c) \times F(a, b).$$

  Associativity and identity laws for this composition are only up to canonical isomorphism. For this reason Dist is a bicategory [**?**].
- There is a symmetric monoidal structure on Dist given by the cartesian product of categories: $A \otimes B = A \times B$.
- The bicategory of distributors is compact closed and $A^{\perp} = A^{\mathrm{op}}$. The linear exponential object is then defined as $A \Rightarrow B = A^{\mathrm{op}} \times B$.
- $\mathrm{Dist}(A, B) = \mathrm{Cat}(A^{\mathrm{op}} \times B, \mathrm{Set})$ is a locally small cocomplete category. For $A, B \in \mathrm{Dist}$ the initial object $\bot_{A,B} \in \mathrm{Dist}(A, B)$ is given by the *zero distributor* defined as follows: for all $\langle a, b \rangle \in A \times B$, $\bot_{A,B}(a, b) = \emptyset$.