Why are Proofs Relevant in Proof-Relevant Models?

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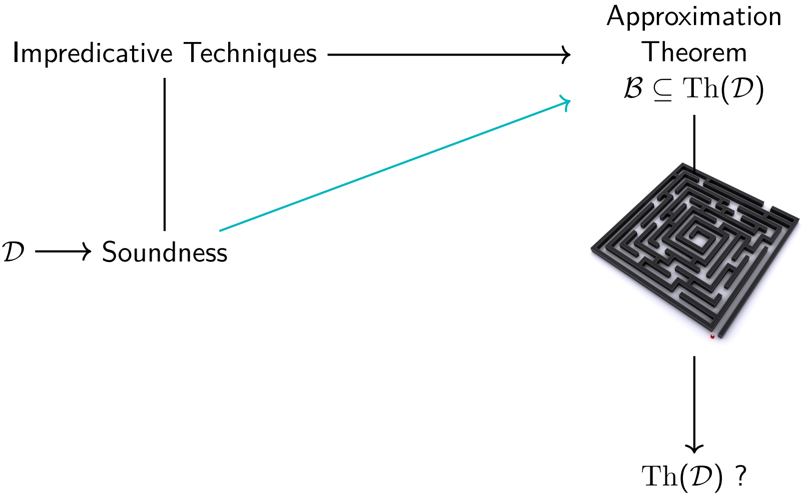
Joint work with F. Olimpieri and G. Manzonetto

Scott Semantics

Impredicative Techniques ⟶ Approximation Theorem $\mathcal{B} \subseteq \mathrm{Th}(\mathcal{D})$

$\mathcal{D} \longrightarrow$ Soundness

$\mathrm{Th}(\mathcal{D})$?


Impredicative Techniques
Approximation Theorem
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Impredicative Techniques $\longrightarrow$ Approximation Theorem $\mathcal{B} \subseteq \mathrm{Th}(\mathcal{D})$

$\mathcal{D} \longrightarrow$ Soundness up to $\cong$

Bicategorical Approximation Theorem $\longrightarrow$ $\mathrm{Th}(\mathcal{D})$?

Approximation Theorem

$$[\![M]\!] = \bigsqcup_{A \in \mathcal{A}(M)} [\![A]\!]$$

Böhm tree of M:

- If M is not head-normalizable, then

$$\mathrm{BT}(M) = \bot,$$

- Otherwise $M \twoheadrightarrow_h \lambda x_1 \ldots x_n.y\, M_1 \cdots M_k$ and

$$\mathrm{BT}(M) = \lambda x_1 \ldots x_n.y$$

$$\mathrm{BT}(M_1) \quad \cdots \quad \mathrm{BT}(M_k)$$

Böhm Trees

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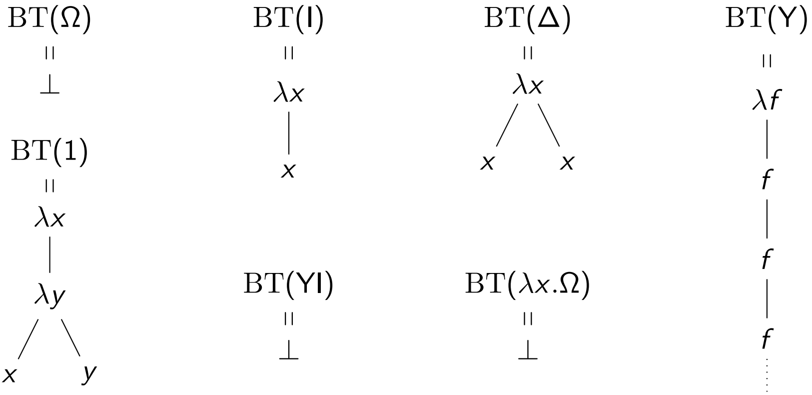
The Böhm Tree Semantics

$$\mathcal{B} \vdash M = N \iff \mathrm{BT}(M) = \mathrm{BT}(N)$$

Examples

$$\mathsf{I} = \lambda x.x \qquad 1 = \lambda xy.xy \qquad \Delta = \lambda x.xx$$

$$\Omega = \Delta\Delta \qquad \mathsf{Y} = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$$



Approximants

$$\Lambda_\bot : \qquad L, O ::= \bot \mid x \mid \lambda x.M \mid MN$$

$\leq_\bot$ least compatible preorder s.t. $\forall L \in \Lambda_\bot, \bot \leq L$

$$\mathcal{A} : \qquad A, B ::= \bot \mid \lambda x_1 \ldots x_n . y A_1 \cdots A_k \quad (\text{for } n, k \geq 0)$$

Approximants of a λ-term

$$\mathcal{A}(M) = \{A \in \mathcal{A} \mid \exists N \in \Lambda,\ M \twoheadrightarrow_\beta N \text{ and } A \leq_\bot N\}$$

Approximants and Böhm Tree

$$\mathrm{BT}(M) = \bigsqcup \mathcal{A}(M)$$

Approximants

$$\Lambda_\perp : \qquad L, O ::= \perp \mid x \mid \lambda x.M \mid MN$$

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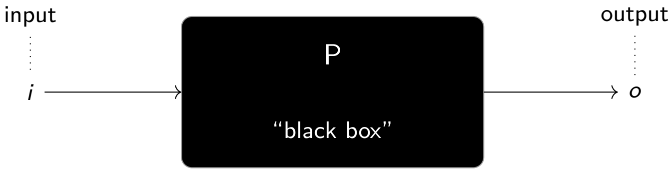
Approximants and Böhm Tree

$$\mathrm{BT}(M) = \bigsqcup \mathcal{A}(M)$$

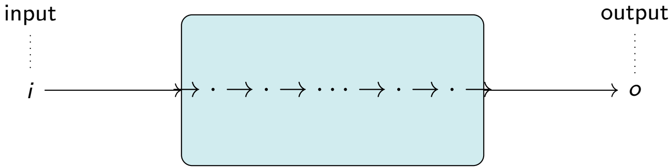
Scott Semantics

A Program $\Gamma \vdash M : A$ is a continous map $[\![M]\!] : [\![\Gamma]\!] \to [\![A]\!]$.



Quantitative Semantics

- Number of steps to termination,
- Amount of resources used during the computation,
- Non-deterministic setting: number of "ways" to get the output.



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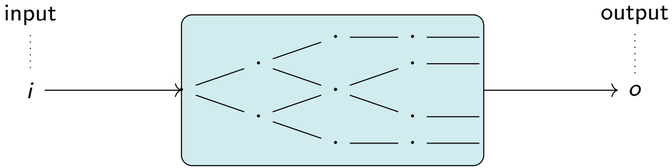

A Syntax for Semantics

Intersection Types (Coppo-Dezani 1980)

$$a, b ::= o \mid a \multimap b \mid (a_1 \cap \cdots \cap a_k)$$

	Filter Models	Graph Models	Relational Models
Idempotency of $\cap$	yes	yes	no
Subtyping	yes	no	no

$$[\![P]\!] = \{(\Gamma, a) \mid \Gamma \vdash P : a\}$$

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Relational Type System

$$\alpha, \beta ::= a \mid \sigma \multimap \alpha \qquad \sigma ::= [\alpha_1, \ldots, \alpha_n]$$

$$\frac{}{x : [\alpha] \vdash x : \alpha} \qquad \frac{\Gamma, x : \sigma \vdash M : \alpha}{\Gamma \vdash \lambda x.M : \sigma \multimap \alpha}$$

$$\frac{\Gamma_0 \vdash M : [\alpha_1, \ldots, \alpha_n] \multimap \alpha \quad \Gamma_1 \vdash N : \alpha_1 \quad \cdots \quad \Gamma_n \vdash N : \alpha_n}{\sum_{i=0}^{n} \Gamma_i \vdash MN : \alpha}$$

Example

$$\vdash \lambda x_1 x_2.M : [\alpha] \multimap [\beta_1, \beta_2] \multimap \alpha'$$

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Categorification

Set-Theoretic	Category-Theoretic
sets	categories
functions	functors
equations	(natural) isomorphisms

A bicategorical model $\mathcal{D}$

$\mathcal{D} = \langle D, \alpha, i, j \rangle$ is a pseudo-reflexive object in a Cartesian Closed Bicategory **C**.

Interpretation of a λ-term M: $[\![M]\!]_{x_1,\dots,x_n} : D^{\& n} \to D$

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\begin{aligned}
[\![x_i]\!]_{x_1,\dots,x_n} &= \pi_i^n, \\
[\![\lambda y.M]\!]_{x_1,\dots,x_n} &= i \circ \lambda([\![M]\!]_{x_1,\dots,x_n,y}), \\
[\![MN]\!]_{x_1,\dots,x_n} &= ev_{D,D} \circ \langle j \circ [\![M]\!]_{x_1,\dots,x_n}, [\![N]\!]_{x_1,\dots,x_n} \rangle.
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Soundness

Theorem of Soundness

$$\text{if } M \to_\beta N \text{ then } [\![M \to_\beta N]\!]_{\vec{x}} : [\![M]\!]_{\vec{x}} \cong [\![N]\!]_{\vec{x}}$$

$$[\![M \to_\beta N]\!]_{\vec{x}}(\Delta, a) : [\![M]\!]_{\vec{x}}(\Delta, a) \cong [\![N]\!]_{\vec{x}}(\Delta, a)$$

Semantic sound with respect to confluence

$$[\![L \twoheadrightarrow_\beta N]\!]_{\vec{x}} \star [\![M \twoheadrightarrow_\beta L]\!]_{\vec{x}} = [\![L' \twoheadrightarrow_\beta N]\!]_{\vec{x}} \star [\![M \twoheadrightarrow_\beta L']\!]_{\vec{x}}$$

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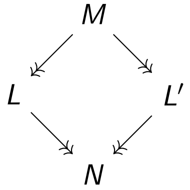

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Bicategorical Models living in CatSym

From relations

$$R : A \times B \to \mathrm{Bool}$$

to distributors

$$F : A^{\mathrm{op}} \times B \to \mathrm{Set}$$

Bicategory of symmetric categorical sequences

- Relational
- $\cap$ as *multisets*
- standard subtyping
- proof-irrelevant and "static" semantics

- Distributors
- $\cap$ as *lists*
- categorical subtyping
- proof-relevant and dynamic semantics

Relational Graph Models

Manzonetto & Ruoppolo'14

A *relational graph model* is a set U with an injection $\iota : \mathrm{M_f}(U) \times U \hookrightarrow U$.

Intersection type presentation:

$$(a_1 \cap \cdots \cap a_k) \multimap a := \iota([a_1, \ldots, a_k], a)$$

Theorem (Breuvart, Manzonetto, Ruoppolo)

$$\mathrm{BT}(M) = \mathrm{BT}(N) \iff [\![M]\!]^U = [\![N]\!]^U, \ \textit{for some } U$$

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Categorified Graph Models

Definition

A *categorified graph model* is a category D and an embedding $\iota : !D^{\mathrm{op}} \times D \hookrightarrow D$.

- Intersection type presentation:

$$\langle a_1, \ldots, a_k \rangle \multimap a := \iota(\langle a_1, \ldots, a_k \rangle, a).$$

- $\langle a_1, \ldots, a_k \rangle$ lives in the *category of lists* $!D$ on D.

Morphisms between Types

Subtypings are generated by *allowable operations* on resources.

$$\sigma : (\langle a_1, \ldots, a_k \rangle \multimap a) \cong (\langle a_{\sigma(1)}, \ldots, a_{\sigma(k)} \rangle \multimap a)$$

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Categorifying Intersection Types

System $R_{\to}$

$$\frac{f : a' \to a}{x_1 : \langle\rangle, \ldots, x_i : \langle a' \rangle, \ldots, x_n : \langle\rangle \vdash x_i : a}$$

$$\frac{\Gamma, x : \vec{a} \vdash M : a \qquad f : (\vec{a} \multimap a) \to b}{\Gamma \vdash \lambda x.M : b}$$

$$\frac{\Gamma_0 \vdash M : \langle a_1, \ldots, a_k \rangle \multimap a \qquad (\Gamma_i \vdash N : a_i)_{i \in [k]}}{\Delta \vdash MN : a}$$

where $\eta : \Delta \to \sum_{j=0}^{k} \Gamma_j$

Congruence

$$\dfrac{\begin{array}{c}\pi_0\\ \vdots \\ \Gamma_0 \vdash \vec{b} \multimap a\end{array} \quad \left(\begin{array}{c}[f_i]\pi_{\sigma^{-1}(i)}\\ \vdots \\ \Gamma_{\sigma^{-1}(i)} \vdash b_i\end{array}\right)_{i=1}^{k} \quad (1 \otimes (\sigma^{-1})^{\star}) \circ \eta}{\Delta \vdash a} \sim \dfrac{\begin{array}{c}[\langle \sigma, \vec{f} \rangle \multimap 1]\pi_0\\ \vdots \\ \Gamma_0 \vdash \vec{a} \multimap a\end{array} \quad \left(\begin{array}{c}\pi_i\\ \vdots \\ \Gamma_i \vdash a_i\end{array}\right)_{i=1}^{k} \quad \eta}{\Delta \vdash a}$$

where $\langle \sigma, f_1, \ldots, f_k \rangle : \vec{a} = \langle a_1, \ldots, a_k \rangle \to \vec{b} = \langle b_1, \ldots, b_k \rangle$, $\langle \sigma, \vec{g} \rangle : \vec{a}' \to \vec{a}$, $g : a \to a'$ and $\theta_i : \Gamma_i \to \Gamma_i'$

Example

Let $k \in \mathbb{N}$, $\sigma \in \mathfrak{S}_k$ and $\pi = \dfrac{\begin{array}{c}\pi_0\\ \vdots \\ \Gamma_0 \vdash \langle a_1, \ldots, a_k \rangle \multimap a\end{array} \quad \left(\begin{array}{c}\pi_i\\ \vdots \\ \Gamma_i \vdash a_i\end{array}\right)_{i=1}^{k} \quad \eta}{\Delta \vdash a}$

Let $\pi' = \dfrac{\begin{array}{c}\pi_0[\sigma \multimap a]\\ \vdots \\ \Gamma_0 \vdash \langle a_{\sigma(1)}, \ldots, a_{\sigma(k)} \rangle \multimap a\end{array} \quad \left(\begin{array}{c}\pi_{\sigma(i)}\\ \vdots \\ \Gamma_{\sigma(i)} \vdash a_{\sigma(i)}\end{array}\right)_{i=1}^{k} \quad \eta'}{\Delta \vdash a}$

and $\eta' = (1 \otimes (\sigma)^{\star}) \circ \eta$

Intersection Type Distributor

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$$\mathsf{T}^D_{\vec{x}}(M) : (\overbrace{!D \times \cdots \times !D}^{\text{len}(\vec{x}) \text{ times}})^{\mathrm{op}} \times D \to \mathrm{Set}$$

$$\mathsf{T}^D_{\vec{x}}(M)(\Delta, a) = \left\{ \begin{array}{c} \tilde{\pi} \\ \vdots \\ \vec{x} : \Delta \vdash M : a \end{array} \right\}$$

$$\mathsf{itd}^M_{\vec{x}} : \mathsf{T}^D_{\vec{x}}(M) \cong [\![M]\!]_{\vec{x}} \circ_{\mathrm{Dist}} \overline{\mu}_1$$

$$\mu_1 : !A_1 \times \cdots \times !A_n \to !(A_1 \sqcup \cdots \sqcup A_n)$$

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Extend Notions to Böhm Trees

$$\mathsf{T}^{D}_{\vec{x}}(\bot) = \emptyset_{!D^n,D} \qquad [\![\bot]\!]_{\vec{x}} = \bot_{D^{\& n},D}$$

Lemma: If $L \leq_\bot P$ then $[\![L]\!]_{\vec{x}} \subseteq [\![P]\!]_{\vec{x}}$ and $\mathsf{T}^{D}_{\vec{x}}(L) \subseteq \mathsf{T}^{D}_{\vec{x}}(P)$.

Consider $\langle \mathcal{A}(M), \leq_\bot \rangle$,

$$\begin{aligned} [\![-]\!]_{\vec{x}} : \mathcal{A}(M) &\to \mathrm{Dist}(!(D^{\& n}), D) \\ A &\mapsto [\![A]\!]_{\vec{x}}, \\ A \leq_\bot A' &\mapsto [\![A]\!]_{\vec{x}} \subseteq [\![A']\!]_{\vec{x}}. \end{aligned}$$

Interpretation of the Böhm Tree

$$[\![\mathrm{BT}(M)]\!]_{\vec{x}} = lim_{A \in \mathcal{A}(M)} [\![A]\!]_{\vec{x}}$$

$$[\![\mathrm{BT}(M)]\!]_{\vec{x}} \circ_{\mathrm{Dist}} \overline{\mu}_1 \cong \mathsf{T}^{D}_{\vec{x}}(\mathrm{BT}(M)).$$

Extend Notions to Böhm Trees

$$\mathsf{T}^D_{\vec{x}}(\perp) = \emptyset_{!D^n,D} \qquad [\![\perp]\!]_{\vec{x}} = \perp_{D^{\& n},D}$$

Lemma: If $L \leq_\perp P$ then $[\![L]\!]_{\vec{x}} \subseteq [\![P]\!]_{\vec{x}}$ and $\mathsf{T}^D_{\vec{x}}(L) \subseteq \mathsf{T}^D_{\vec{x}}(P)$.

Consider $\langle \mathcal{A}(M), \leq_\perp \rangle$,

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Typed Redexes

Only some redexes are typed in derivations.

Example

$$\pi = \dfrac{\dfrac{f : a \to a'}{x : \langle\langle\rangle \multimap a\rangle \vdash x : \langle\rangle \multimap a'}}{x : \langle\langle\rangle \multimap a\rangle \vdash x(\mathsf{II})} \qquad \mathrm{tocc}(\pi) = \{[], [](\mathsf{II})\}$$

The redex $\mathsf{II} = (\lambda x.x)(\lambda x.x)$ is not typed in the derivation π.

Typed Reductions

Normalization along M

$\tilde{\pi} \in \mathrm{T}^{D}_{\vec{x}}(M)(\Delta, a)$ is normalizable along M if $\exists N \in \Lambda, M \twoheadrightarrow_\beta N$ and $\mathrm{T}^{D}_{\vec{x}}(M \twoheadrightarrow_\beta N)_{\Delta,a}(\tilde{\pi})$ is in normal form.

The reduction strategy contracting typed redexes in type derivations along M is strongly normalizing.

$$\mathrm{nf}(\mathrm{T}^{D}_{\vec{x}}(M)(\Delta, a)) = \{\mathrm{nf}(\tilde{\pi}) \in R_{\to} \mid \tilde{\pi} \in [\![M]\!]_{\vec{x}}(\Delta, a)\}$$

Normalization Theorem

$$\mathrm{Norm}_{\vec{x}}(M) : \mathrm{T}^{D}_{\vec{x}}(M) \cong \mathrm{nf}(\mathrm{T}^{D}_{\vec{x}}(M))$$

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Minimal Terms

Define a map $L^{\vec{x}}_{-} : R_{\rightarrow} \rightarrow \Lambda_{\perp}$ by induction on the structure of π as follows:

- if $\pi = \dfrac{f : a' \rightarrow a}{x_1 : \langle\rangle, \ldots, x_i : \langle a' \rangle, \ldots, x_n : \langle\rangle \vdash x_i : a}\ \text{ax}$ then $L^{\vec{x}}_{\pi} = x_i$;
- if $\pi = \dfrac{\dfrac{\pi'}{\Delta, x : \vec{a} \vdash M : a} \quad f : (\vec{a} \multimap a) \rightarrow b}{\Delta \vdash \lambda x.M : b}\ \text{abs}$ then $L^{\vec{x}}_{\pi} = \lambda y.(L^{\vec{x},y}_{\pi'})$;
- if $\pi =$

$$\dfrac{\dfrac{\pi_0}{\Gamma_0 \vdash M : \langle a_1, \ldots, a_k \rangle \multimap a} \quad \dfrac{\pi_i}{(\Gamma_i \vdash N : a_i)^k_{i=1}} \quad \eta : \Delta \rightarrow \bigotimes^k_{j=0} \Gamma_j}{\Delta \vdash MN : a}\ \text{app}$$

  then $L^{\vec{x}}_{\pi} = L^{\vec{x}}_{\pi_0}(\bigvee^k_{i=1} L^{\vec{x}}_{\pi_i})$.

Examples

Let $\pi_1 = \dfrac{\dfrac{f : a' \to a}{x : \langle\langle\rangle \multimap a'\rangle \vdash x : \langle\rangle \multimap a}}{x : \langle\langle\rangle \multimap a'\rangle \vdash x\Omega : a}$ $\qquad L^{x}_{\pi_1} = x\bot$

Let $\pi_2 =$

$$\dfrac{\dfrac{}{x : \langle\langle a, a\rangle \multimap a\rangle \vdash x : \langle a, a\rangle \multimap a} \quad \dfrac{\dfrac{}{y : \langle\langle\rangle \multimap a\rangle \vdash y : \langle\rangle \multimap a}}{y : \langle\langle\rangle \multimap a\rangle \vdash yz : a} \quad \dfrac{\dfrac{}{y : \langle\langle a\rangle \multimap a\rangle \vdash y : \langle a\rangle \multimap a} \quad \dfrac{}{z : \langle a\rangle \vdash z : a}}{y : \langle\langle a\rangle \multimap a\rangle, z : \langle a\rangle \vdash yz : a}}{x : \langle\langle a, a\rangle \multimap a\rangle, y : \langle\langle a\rangle \multimap a, \langle\rangle \multimap a\rangle, z : \langle a\rangle \vdash x(yz) : a}$$

$L^{\langle x,y,z\rangle}_{\pi_2} = x(yz)$

Approximation Theorem

Proposition: Let $M \in \Lambda^o(\vec{x})$ and $\pi \in R_\rightarrow(M)$.

- $\pi \in R_\rightarrow(L_\pi^{\vec{x}})$ and $L_\pi^{\vec{x}} \leq_\perp M$.
- If π is a normal form then $L_\pi^{\vec{x}} \in \mathcal{A}$, whence $L_\pi^{\vec{x}} \in \mathcal{A}(M)$.

Commutation Theorem:

$$\mathsf{nf}(\mathsf{T}_{\vec{x}}^{D}(M)) = \mathsf{T}_{\vec{x}}^{D}(\mathrm{BT}(M))$$

Approximation Theorem

$$\mathsf{appr}_{\vec{x}}(M) : \mathsf{T}_{\vec{x}}^{D}(M) \cong \mathsf{T}_{\vec{x}}^{D}(\mathrm{BT}(M))$$

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Theory of a Bicategorical Model

$\alpha : [\![M]\!]_{\vec{x}} \cong [\![N]\!]_{\vec{x}}$ coherent wrt β-normalization when the induced natural isomorphism $\alpha : \mathsf{T}^{D}_{\vec{x}}(M) \cong \mathsf{T}^{D}_{\vec{x}}(N)$ satisfies: $\forall \tilde{\pi} \in \mathsf{T}^{D}_{\vec{x}}(M)(\Delta, a)$ we have $\mathsf{nf}(\tilde{\pi}) = \mathsf{nf}(\alpha_{\Delta,a}(\tilde{\pi}))$

$$\mathrm{Th}(\mathcal{D}) = \{(M, N) \mid \begin{array}{l} \alpha : [\![M]\!]_{\vec{x}} \cong [\![N]\!]_{\vec{x}} \\ \alpha \textit{ coherent wrt } \beta\textit{-normalization}\} \end{array}$$

Characterization of the Theory

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$$\mathsf{T}^D_{\vec{x}}(M) \cong \mathsf{T}^D_{\vec{x}}(N) \textit{ iff } \mathrm{BT}(M) = \mathrm{BT}(N)$$

($\Leftarrow$) By Approximation Theorem.
($\Rightarrow$) Assume $\mathsf{T}^D_{\vec{x}}(M) \cong \mathsf{T}^D_{\vec{x}}(N)$ and $\mathrm{BT}(M) \neq \mathrm{BT}(N)$, towards a contradiction:

- there is some $A \in \mathcal{A}(M) \setminus \mathcal{A}(N)$,
- so there is $\tilde{\pi} \in |\mathsf{nf}(\mathsf{T}^D_{\vec{x}}(M))| = |\mathsf{nf}(\mathsf{T}^D_{\vec{x}}(N))|$ such that $A^{\vec{x}}_{\tilde{\pi}} = P$,
- and by definition of normalization along N, $\tilde{\pi} \in |\mathsf{T}^D_{\vec{x}}(N')|$ for some N' such that $N \twoheadrightarrow_\beta N'$.
- We obtain $A^{\vec{x}}_{\tilde{\pi}} = P \leq_\perp N'$, so $P \in \mathcal{A}(N)$. Contradiction.

$$\mathrm{Th}(\mathcal{D}) = \mathcal{B}$$

Perspectives

- Developing a theory for 2-dimensional λ-theories.
- Considering models from different kind of intersection type constructions.

Merci de votre attention!

Decategorification

Polr category of preorders and monotonic relations

decategorification of Dist to Polr

1. small category A: $|\mathsf{Dec}(A)| = A$ and $a \leq_{\mathsf{Dec}A} b$ whenever $A(a,b) \neq \emptyset$.
2. small categories A and B, $F : A \nrightarrow B$

$$\mathsf{Dec}_{A,B}(F) = \{\langle a, b\rangle \in |\mathsf{Dec}(A)^{op} \times \mathsf{Dec}(B)| \mid F(a,b) \neq \emptyset\}.$$

$$\mathsf{Dec}(\mathsf{T}^{D}_{\vec{x}}(M)) = [\![M]\!]^{\mathrm{MPolr}}_{\vec{x}}$$

Approximation Theorem: $[\![M]\!]^{\mathrm{MPolr}}_{\vec{x}} = [\![\mathrm{BT}(M)]\!]^{\mathrm{MPolr}}_{\vec{x}}$

$$\mathcal{B} = \mathrm{Th}(D_A) \subseteq \mathrm{Th}(U_{\mathsf{Dec}(A)})$$

Commutation Theorem

For all $M \in \Lambda^o(\vec{x})$,

$$\mathsf{nf}(\mathsf{T}^D_{\vec{x}}(M)) = \mathsf{T}^D_{\vec{x}}(\mathrm{BT}(M)).$$

Proof: $(\subseteq)$ Let $\tilde{\pi} \in \mathsf{nf}(\mathsf{T}^D_{\vec{x}}(M))(\Delta, a)$. By definition of normalization along M, there exists $\tilde{\rho} \in \mathsf{T}^D_{\vec{x}}(M)(\Delta, a)$ and $N \in \Lambda$ such that $\tilde{\pi} = \mathsf{nf}(\tilde{\rho})$ and $\tilde{\pi} \in \mathsf{T}^D_{\vec{x}}(N)(\Delta, a)$ with $M \twoheadrightarrow_\beta N$. By previous proposition, we get $\tilde{\pi} \in \mathsf{T}^D_{\vec{x}}(A^{\vec{x}}_\pi)$ and $A^{\vec{x}}_\pi \leq_\perp N$ is a $\beta\perp$-nf. Thus $A^{\vec{x}}_\pi \in \mathcal{A}(N)$, so we conclude $\tilde{\pi} \in \mathsf{T}^D_{\vec{x}}(\mathrm{BT}(M))(\Delta, a)$.
$(\supseteq)$ Let $\tilde{\pi} \in \mathrm{BT}(M)(\Delta, a)$. By definition, there exists a $P \in \mathcal{A}(M)$ such that $\tilde{\pi} \in \mathsf{T}^D_{\vec{x}}(P)(\Delta, a)$. Such a $\tilde{\pi}$ is a normal form. By Lemma Inclusion of Interpreations and the definition of $\mathcal{A}(M)$, we get $\mathsf{T}^D_{\vec{x}}(P) \subseteq \mathsf{T}^D_{\vec{x}}(N)$ for some N such that $M \twoheadrightarrow_\beta N$. By Theorem Soundness, we conclude that there exists $\tilde{\rho} \in \mathsf{T}^D_{\vec{x}}(M)$ such that $\tilde{\pi}$ is the normal form of $\tilde{\rho}$.

$$\left(\frac{f : a' \to a}{\langle\rangle, \ldots, \langle a' \rangle, \ldots, \langle\rangle \vdash a}\right)\{g : b \to a'\} \quad = \quad \frac{f \circ g}{\langle\rangle, \ldots, \langle b \rangle, \ldots, \langle\rangle \vdash a}$$

$$\left(\frac{\begin{array}{c}\pi \\ \vdots \\ \Delta, \vec{a} \vdash a\end{array} \qquad f : (\vec{a} \multimap a) \to b}{\Delta \vdash b}\right)\{\eta\} \quad = \quad \frac{\begin{array}{c}\pi\{\eta \oplus \langle 1 \rangle\} \\ \vdots \\ \Delta', \vec{a} \vdash a\end{array} \qquad f : (\vec{a} \multimap a) \to b}{\Delta' \vdash \vec{a} \multimap a}$$

$$\left(\frac{\begin{array}{c}\pi_1 \\ \vdots \\ \Gamma_0 \vdash \vec{a} \multimap a\end{array} \qquad \left(\begin{array}{c}\pi_i \\ \vdots \\ \Gamma_i \vdash a_i\end{array}\right)_{i=1}^{k} \qquad \theta : \Delta \to \bigotimes_{j=0}^{k} \Gamma_j}{\Delta \vdash a}\right)\{\eta\} = \frac{\begin{array}{c}\pi_1 \\ \vdots \\ \Gamma_0 \vdash \vec{a} \multimap a\end{array} \qquad \left(\begin{array}{c}\pi_i \\ \vdots \\ \Gamma_i \vdash a_i\end{array}\right)_{i=1}^{k} \qquad \theta \circ \eta}{\Delta' \vdash a}$$

where $\vec{a} = \langle a_1, \ldots, a_k \rangle$ and $\eta : \Delta' \to \Delta$.

Figure: Right action on derivations.

Bicategory $\mathbf{C}$

- objects $A, B \in Ob(\mathbf{C})$ also called 0-cells;
- for all $A, B \in \mathbf{C}$, a category $\mathbf{C}(A, B)$;
  objects in $\mathbf{C}(A, B)$ named 1-cells or morphisms from A to B;
  arrows in $\mathbf{C}(A, B)$ (between 1-cells) named 2-cells;
  composition of 2-cells called vertical composition;
- for every $A, B, C \in \mathbf{C}$, a bifunctor called horizontal composition
  $\circ_{A,B,C} \colon \mathbf{C}(A, C) \times \mathbf{C}(A, B) \to \mathbf{C}(A, C)$;
- for every $A \in \mathbf{C}$, a functor $1_A \colon 1 \to \mathbf{C}(A, A)$;
- for all 1-cells $F \colon A \to B$, $G \colon B \to C$, and $H \colon C \to D$, a family of invertible 2-cells expressing the associativity law
  $\alpha_{H,G,F} \colon H \circ (G \circ F) \cong (H \circ G) \circ F$;
- for every 1-cell $F \colon A \to B$, two families of invertible 2-cells expressing the identity law
  $\lambda_F \colon 1_B \circ F \cong F, \quad \rho_F \colon F \cong F \circ 1_A.$

Bicategory of Distributors

- 0-cells are small categories $A, B, C, \ldots$
- 1 cells $F : A \nrightarrow B$ are functors $F : A^{\mathrm{op}} \times B \to \mathrm{Set}$.
- 2-cells $\alpha : F \Rightarrow G$ are natural transformations.
- For fixed 0-cells A and B, the 1-cells and 2-cells are organized as a category $\mathrm{Dist}(A, B)$.
- For $A \in \mathrm{Dist}$, the identity $1_A : A \nrightarrow A$ is defined as the Yoneda embedding $1_A(a, a') = A(a, a')$.
- For 1-cells $F : A \nrightarrow B$ and $G : B \nrightarrow C$, the *horizontal composition* is given by

$$(G \circ F)(a, c) = \int^{b \in B} G(b, c) \times F(a, b).$$

  Associativity and identity laws for this composition are only up to canonical isomorphism. For this reason Dist is a bicategory [**borc:cat**].
- There is a symmetric monoidal structure on Dist given by the cartesian product of categories: $A \otimes B = A \times B$.
- The bicategory of distributors is compact closed and $A^{\perp} = A^{\mathrm{op}}$. The linear exponential object is then defined as $A \Rightarrow B = A^{\mathrm{op}} \times B$.
- $\mathrm{Dist}(A, B) = \mathrm{Cat}(A^{\mathrm{op}} \times B, \mathrm{Set})$ is a locally small cocomplete category. For $A, B \in \mathrm{Dist}$ the initial object $\perp_{A,B} \in \mathrm{Dist}(A, B)$ is given by the *zero distributor* defined as follows: for all $\langle a, b \rangle \in A \times B, \perp_{A,B}(a, b) = \emptyset$.

Symmetric strict monoidal completion

Let A be a small category. The symmetric strict monoidal completion $!A$ of A is the category:

- $!A = \{\langle a_1, \ldots, a_n \rangle \mid a_i \in A \text{ and } n \in \mathbb{N}\}$;
- $!A[\langle a_1, \ldots, a_n \rangle, \langle a'_1, \ldots, a'_{n'} \rangle] = \begin{cases} \{\langle \sigma, f_i \rangle_{i \in [n]} \mid f_i : a_i \to a'_{\sigma(i)},\ \sigma \in S_n\}, \text{if } n = n'; \\ \emptyset, \text{otherwise}; \end{cases}$
- for $f = \langle \sigma, f_i \rangle_{i \in [n]} \colon \vec{a} \to \vec{b}$ and $g = \langle \tau, g_i \rangle_{i \in [n]} \colon \vec{b} \to \vec{c}$ their composition is defined as follows $g \circ f = \langle \tau\sigma, g_{\sigma(1)} \circ f_1, \ldots, g_{\sigma(n)} \circ f_n \rangle$;
- for $\vec{a} = \langle a_1, \ldots, a_n \rangle \in !A$, the identity on $\vec{a}$ is given by $1_{\vec{a}} = \langle 1_n, 1_{a_1}, \ldots, 1_{a_n} \rangle$;
- the monoidal structure $\vec{a} \oplus \vec{b}$ is given by list concatenation.

CatSym

The endofunctor $! : \mathrm{Cat} \to \mathrm{Cat}$, can be lifted to a pseudocomonad over Dist, we denote as CatSym its Klesli bicategory:

- $Ob(\mathrm{CatSym})$ are the small categories
- For $A, B \in \mathrm{CatSym}$, we have $\mathrm{CatSym}(A, B) = Dist(B, !A)$.
- The identity $1_A(\vec{a}, a) = !A(\vec{a}, \langle a \rangle)$.
- For $F : A \nrightarrow B$ and $G : B \nrightarrow C$, composition is given by $(G \circ F)(a, c) = \int^{b \in !B} G(b, c) \times F(a, b)$.
- CatSym is cartesian, with cartesian product the disjoint union $A \& B = A \bigsqcup B$. The terminal object is the empty category.
- CatSym is cartesian closed, with exponential object $A \multimap B = !A^{\mathrm{op}} \times B$.

Pseudo-reflexive Object

Given a small category A, we define an inductive family of small categories:

$$D_0 = A, \qquad D_{n+1} = (!D_n^{op} \times D_n) \sqcup A.$$

Then, we construct a family of functors $\iota_n : D_n \hookrightarrow D_{n+1}$, again by induction:

$$\iota_0 = \mathrm{in}_A, \qquad \iota_{n+1} = (!(\iota_n)^{op} \times \iota_n) \sqcup 1_A.$$

Directed colimit $D_A = lim_{n \in \mathbb{N}} D_n$
Free algebra $\langle D_A, \iota :!D_A^{op} \times D_A \to D_A \rangle$ with a retraction pair $D_A \Rightarrow D_A \lhd D_A$