Towards Categorical Structures for Operational Game Semantics
(a Work in Progress)

Axel Kerinec
joint work with Beniamino Accattoli, Guilhem Jaber and Adrienne Lancelot

June 5, 2024

What is an OGS?

Operational Game Semantics:

a semantic based on the analysis of normal forms
to deduce the interactions with the environment.

What is an OGS?

Operational Game Semantics:

a semantic based on the analysis of normal forms to deduce the interactions with the environment.

Already a close composition: $[\![E[T]]\!] = [\![E]\!] \circ [\![T]\!]$.

What is an OGS?

Operational Game Semantics:

a semantic based on the analysis of normal forms
to deduce the interactions with the environment.

Already a close composition: $[\![E[T]]\!] = [\![E]\!] \circ [\![T]\!]$.
Towards the definition of an open composition: $[\![TU]\!] = [\![T]\!] \circ [\![U]\!]$.

Outline

- A semantic category $\mathcal{C}_{sem}$:

  based on LTSs and parallel composition;

- A syntactic category $\mathcal{C}_{syn}$:

  based on name assignations and the substitution;

- A functor between $\mathcal{C}_{syn}$ and $\mathcal{C}_{sem}$.

The λ-Calculus

Terms: $\mathtt{T}, \mathtt{U} \quad \triangleq \quad x \mid \lambda x.\mathtt{T} \mid \mathtt{TU}$

The λ-Calculus

Terms: $\mathtt{T}, \mathtt{U} \triangleq x \mid \lambda x.\mathtt{T} \mid \mathtt{TU}$

Definition (Reduction)

Evaluation Contexts: $\mathtt{E}, \mathtt{F} \triangleq [\cdot] \mid \mathtt{ET}$

Reduction: $\mathtt{E}[(\lambda x.\mathtt{T})\mathtt{U}] \rightarrow \mathtt{E}[\mathtt{T}\{\mathtt{U}/x\}]$

The λ-Calculus

Terms: $\mathtt{T}, \mathtt{U} \triangleq x \mid \lambda x.\mathtt{T} \mid \mathtt{TU}$

Definition (Reduction)

Evaluation Contexts: $\mathtt{E}, \mathtt{F} \triangleq [\cdot] \mid \mathtt{ET}$

Reduction: $\mathtt{E}[(\lambda x.\mathtt{T})\mathtt{U}] \rightarrow \mathtt{E}[\mathtt{T}\{\mathtt{U}/x\}]$

$$
\begin{array}{rl}
I = \lambda x.x & Ix \rightarrow_\beta x \\
\Omega = (\lambda x.xx)(\lambda x.xx) & \Omega \rightarrow_\beta \Omega \rightarrow_\beta \cdots \rightarrow_\beta \Omega \rightarrow_\beta \cdots \\
X = (\lambda x.(xx)f)(\lambda x.(xx)f) & X \rightarrow (((\lambda x.(xx)f)(\lambda x.(xx)f))f) \\
 & \quad \rightarrow ((((\lambda x.(xx)f)(\lambda x.(xx)f))f)f) \\
 & \quad \rightarrow \cdots \rightarrow ((((\cdots)f)f)f)
\end{array}
$$

Game LTS

Definition (Game bipartite LTS $\mathcal{G} = (\mathrm{Pos}, \mathrm{Moves}, \rightarrow)$)

With ϕ^{in} and ϕ^{in} sets of names a, b

$$\frac{a \in \phi^{\text{out}} \qquad b_1, \ldots, b_k \notin \phi^{\text{in}}}{\langle \phi^{\text{in}} \mid \phi^{\text{out}} \rangle^{\oplus} \xrightarrow{a!(b_1,\ldots,b_k)} \langle \phi^{\text{in}} \cup \{b_1, \ldots, b_k\} \mid \phi^{\text{out}} \rangle^{\ominus}}$$

$$\frac{a \in \phi^{\text{in}} \qquad b_1, \ldots, b_k \notin \phi^{\text{out}}}{\langle \phi^{\text{in}} \mid \phi^{\text{out}} \rangle^{\ominus} \xrightarrow{a?(b_1,\ldots,b_k)} \langle \phi^{\text{in}} \mid \phi^{\text{out}} \cup \{b_1, \ldots, b_k\} \rangle^{\oplus}}$$

A Few Definitions

- An *LTS morphism* from $\mathcal{L}_1 = (\textsc{States}_1, \textsc{Actions}, \rightarrow_1)$ to $\mathcal{L}_2 = (\textsc{States}_2, \textsc{Actions}, \rightarrow_2)$ is a function $f : \textsc{States}_1 \rightarrow \textsc{States}_2$ such that for all transitions $\mathbb{S} \xrightarrow{\textbf{act}}_1 \mathbb{R}$ of $\mathcal{L}_1$, there is $f(\mathbb{S}) \xrightarrow{\textbf{act}}_2 f(\mathbb{R})$ in $\mathcal{L}_2$.

A Few Definitions

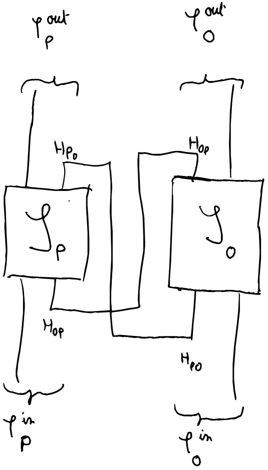
- An *LTS morphism* from $\mathcal{L}_1 = (\textsc{States}_1, \textsc{Actions}, \rightarrow_1)$ to $\mathcal{L}_2 = (\textsc{States}_2, \textsc{Actions}, \rightarrow_2)$ is a function $f : \textsc{States}_1 \rightarrow \textsc{States}_2$ such that for all transitions $\mathbb{S} \xrightarrow{\textbf{act}}_1 \mathbb{R}$ of $\mathcal{L}_1$, there is $f(\mathbb{S}) \xrightarrow{\textbf{act}}_2 f(\mathbb{R})$ in $\mathcal{L}_2$.
- A *game-indexed LTS* is a pair $(\mathcal{L}, \circ\!\!\circ)$ formed by a bipartite LTS, together a bipartite LTS morphism $\circ\!\!\circ$ between $\mathcal{L}$ and the Game LTS $\mathcal{G}$.

A Few Definitions

- An *LTS morphism* from $\mathcal{L}_1 = (\textsc{States}_1, \textsc{Actions}, \rightarrow_1)$ to $\mathcal{L}_2 = (\textsc{States}_2, \textsc{Actions}, \rightarrow_2)$ is a function
  $f : \textsc{States}_1 \rightarrow \textsc{States}_2$ such that for all transitions $\mathbb{S} \xrightarrow{\textbf{act}}_1 \mathbb{R}$ of $\mathcal{L}_1$, there is $f(\mathbb{S}) \xrightarrow{\textbf{act}}_2 f(\mathbb{R})$ in $\mathcal{L}_2$.
- A *game-indexed LTS* is a pair $(\mathcal{L}, \circ\!\!\circ)$ formed by a bipartite LTS, together a bipartite LTS morphism $\circ\!\!\circ$ between $\mathcal{L}$ and the Game LTS $\mathcal{G}$.
- A *strategy* $\mathcal{S} \in \mathrm{Strats}$ is a triple $(\mathcal{L}, \circ\!\!\circ, \mathbb{S})$ formed by a game-indexed LTS $(\mathcal{L}, \circ\!\!\circ)$, and a passive state $\mathbb{S}$.
  We write $\mathrm{Strats}[\mathbb{P}]$ for the strategies $(\mathcal{L}, \circ\!\!\circ, \mathbb{S})$ such that $\mathbb{S} \circ\!\!\circ \mathbb{P}$.

A Few Definitions

- An *LTS morphism* from $\mathcal{L}_1 = (\textsc{States}_1, \textsc{Actions}, \to_1)$ to $\mathcal{L}_2 = (\textsc{States}_2, \textsc{Actions}, \to_2)$ is a function
  $f : \textsc{States}_1 \to \textsc{States}_2$ such that for all transitions $\mathbb{S} \xrightarrow{\textbf{act}}_1 \mathbb{R}$ of $\mathcal{L}_1$, there is $f(\mathbb{S}) \xrightarrow{\textbf{act}}_2 f(\mathbb{R})$ in $\mathcal{L}_2$.
- A *game-indexed LTS* is a pair $(\mathcal{L}, \overset{\circ}{\circ})$ formed by a bipartite LTS, together a bipartite LTS morphism $\overset{\circ}{\circ}$ between $\mathcal{L}$ and the Game LTS $\mathcal{G}$.
- A *strategy* $\mathcal{S} \in \mathrm{Strats}$ is a triple $(\mathcal{L}, \overset{\circ}{\circ}, \mathbb{S})$ formed by a game-indexed LTS $(\mathcal{L}, \overset{\circ}{\circ})$, and a passive state $\mathbb{S}$.
  We write $\mathrm{Strats}[\mathbb{P}]$ for the strategies $(\mathcal{L}, \overset{\circ}{\circ}, \mathbb{S})$ such that $\mathbb{S} \mathrel{\overset{\circ}{\circ}} \mathbb{P}$.
- A game-indexed LTS $(\mathcal{L}, \overset{\circ}{\circ})$ is *receptive* when for all $\mathbb{S} \mathrel{\overset{\circ}{\circ}} \mathbb{P}$ with $\mathbb{P}$ passive, if $\mathbb{P} \xrightarrow{\textbf{m}} \mathbb{Q}$ then there exists a state $\mathbb{R}$ such that $\mathbb{S} \xrightarrow{\textbf{m}} \mathbb{R}$ and $\mathbb{R} \mathrel{\overset{\circ}{\circ}} \mathbb{Q}$.



- states: $\mathbb{S}_P \;{}_{\mathrm{H}_{PO}}||_{\mathrm{H}_{OP}}\; \mathbb{S}_O$ with $\mathrm{H}_{PO}, \mathrm{H}_{OP}$ *hidden names*,
- visible actions: Moves, silent actions: sync,
- transition function: (with $\mathbf{m} = a(b_1, \ldots, b_k)$)

$$\frac{\mathbb{S}_O \xrightarrow{\mathbf{m}}_O \mathbb{R}_O \quad \mathbb{S}_P \text{ passive} \quad a \notin \mathrm{H}}{\mathbb{S}_P \;{}_{\mathrm{H}_{PO}}||_{\mathrm{H}_{OP}}\; \mathbb{S}_O \xrightarrow{\mathbf{m}} \mathbb{S}_P \;{}_{\mathrm{H}_{PO}}||_{\mathrm{H}_{OP}}\; \mathbb{R}_O}$$

$$\frac{\mathbb{S}_P \xrightarrow{\mathbf{m}}_P \mathbb{R}_P \quad \mathbb{S}_O \text{ passive} \quad a \notin \mathrm{H}}{\mathbb{S}_P \;{}_{\mathrm{H}_{PO}}||_{\mathrm{H}_{OP}}\; \mathbb{S}_O \xrightarrow{\mathbf{m}} \mathbb{R}_P \;{}_{\mathrm{H}_{PO}}||_{\mathrm{H}_{OP}}\; \mathbb{S}_O}$$

$$\frac{\mathbb{S}_P \xrightarrow{\mathbf{m}}_P \mathbb{R}_P \quad \mathbb{S}_O \xrightarrow{\overline{\mathbf{m}}}_O \mathbb{R}_O \quad a \in \mathrm{H}_{PO}}{\mathbb{S}_P \;{}_{\mathrm{H}_{PO}}||_{\mathrm{H}_{OP}}\; \mathbb{S}_O \xrightarrow{\text{sync}} \mathbb{R}_P \;{}_{\{b_1,\ldots,b_k\}\cup\mathrm{H}_{PO}}||_{\mathrm{H}_{OP}}\; \mathbb{R}_O}$$

$$\frac{\mathbb{S}_P \xrightarrow{\mathbf{m}}_P \mathbb{R}_P \quad \mathbb{S}_O \xrightarrow{\overline{\mathbf{m}}}_O \mathbb{R}_O \quad a \in \mathrm{H}_{OP}}{\mathbb{S}_P \;{}_{\mathrm{H}_{PO}}||_{\mathrm{H}_{OP}}\; \mathbb{S}_O \xrightarrow{\text{sync}} \mathbb{R}_P \;{}_{\mathrm{H}_{PO}}||_{\{b_1,\ldots,b_k\}\cup\mathrm{H}_{OP}}\; \mathbb{R}_O}$$

Semantic Category

There is an LTS morphism between $\mathcal{G}||\mathcal{G}$ and $\mathcal{G}$.

Semantic Category

There is an LTS morphism between $\mathcal{G}||\mathcal{G}$ and $\mathcal{G}$.

Composition: for $\mathcal{S}_1 \in \mathrm{Strats}[\langle \phi^{\mathsf{in}} \mid \phi \rangle^{\ominus}]$ and $\mathcal{S}_2 \in \mathrm{Strats}[\langle \phi \mid \phi^{\mathsf{out}} \rangle^{\ominus}]$

$$\mathcal{S}_2 \circ \mathcal{S}_1 = \mathcal{S}_1 \,{}_{\phi}||_{\emptyset}\, \mathcal{S}_2 \in \mathrm{Strats}[\langle \phi^{\mathsf{in}} \mid \phi^{\mathsf{out}} \rangle^{\ominus}].$$

Semantic Category

There is an LTS morphism between $\mathcal{G}||\mathcal{G}$ and $\mathcal{G}$.

Composition: for $\mathcal{S}_1 \in \mathrm{Strats}[\langle \phi^{\mathsf{in}} \mid \phi \rangle^{\ominus}]$ and $\mathcal{S}_2 \in \mathrm{Strats}[\langle \phi \mid \phi^{\mathsf{out}} \rangle^{\ominus}]$
$\mathcal{S}_2 \circ \mathcal{S}_1 = \mathcal{S}_1 \,{}_{\phi}||_{\emptyset}\, \mathcal{S}_2 \in \mathrm{Strats}[\langle \phi^{\mathsf{in}} \mid \phi^{\mathsf{out}} \rangle^{\ominus}]$.

Definition ($\mathcal{C}_{sem}$)

- objects: set of names ϕ,
- morphisms between ϕ and ψ: strategies $\mathcal{S} \in \mathrm{Strats}[\langle \phi \mid \psi \rangle^{\ominus}]$ quotiented by bisimilarity,
- composition of two morphisms as above,
- identity morphism over ϕ: the bisimilarity quotient of the Forwarder strategy $\mathcal{F}_{\phi}$.

Syntactic Category

Names: $a = x \mid v \mid c$
Name assignations: partial maps s.t. $\gamma(x) = \mathtt{T}, \gamma(v) = \mathtt{V}$ and $\gamma(c) = [d]E$

Syntactic Category

Names: $a = x \mid v \mid c$
Name assignations: partial maps s.t. $\gamma(x) = \mathtt{T}, \gamma(v) = \mathtt{V}$ and $\gamma(c) = [d]E$

We write $\phi^{\mathsf{in}} \vdash \gamma : \phi^{\mathsf{out}}$ *when* $\forall a \in \phi^{\mathsf{out}}$*:* $\mathsf{supp}(\gamma(a)) \subseteq \phi^{\mathsf{in}}$*.*

Syntactic Category

Names: $a = x \mid v \mid c$
Name assignations: partial maps s.t. $\gamma(x) = \mathtt{T}, \gamma(v) = \mathtt{V}$ and $\gamma(c) = [d]E$

We write $\phi^{\mathrm{in}} \vdash \gamma : \phi^{\mathrm{out}}$ *when* $\forall a \in \phi^{\mathrm{out}}$*:* $\mathrm{supp}(\gamma(a)) \subseteq \phi^{\mathrm{in}}$*.*

Definition (syntactic category $\mathcal{C}_{syn}$)

- objects are sets of names ϕ,

Syntactic Category

Names: $a = x \mid v \mid c$
Name assignations: partial maps s.t. $\gamma(x) = \mathtt{T}, \gamma(v) = \mathtt{V}$ and $\gamma(c) = [d]E$

We write $\phi^{\text{in}} \vdash \gamma : \phi^{\text{out}}$ *when* $\forall a \in \phi^{\text{out}}$*:* $\text{supp}(\gamma(a)) \subseteq \phi^{\text{in}}$*.*

Definition (syntactic category $\mathcal{C}_{syn}$)

- objects are sets of names ϕ,
- morphisms are name assignations,

Syntactic Category

Names: $a = x \mid v \mid c$
Name assignations: partial maps s.t. $\gamma(x) = \mathtt{T}, \gamma(v) = \mathtt{V}$ and $\gamma(c) = [d]E$

We write $\phi^{\mathsf{in}} \vdash \gamma : \phi^{\mathsf{out}}$ *when* $\forall a \in \phi^{\mathsf{out}}$*:* $\mathsf{supp}(\gamma(a)) \subseteq \phi^{\mathsf{in}}$*.*

Definition (syntactic category $\mathcal{C}_{syn}$)

- objects are sets of names ϕ,
- morphisms are name assignations,
- composition of $\phi^{\mathsf{in}} \vdash \gamma : \phi$ and $\phi \vdash \delta : \phi^{\mathsf{out}}$: $\gamma \circ \delta$ is the partial map from $a \in \phi^{\mathsf{out}}$ to $\delta(a)\{\gamma\}$. We then have $\phi^{\mathsf{in}} \vdash \gamma \circ \delta : \phi^{\mathsf{out}}$,

Syntactic Category

Names: $a = x \mid v \mid c$
Name assignations: partial maps s.t. $\gamma(x) = \mathtt{T}, \gamma(v) = \mathtt{V}$ and $\gamma(c) = [d]E$

We write $\phi^{\mathsf{in}} \vdash \gamma : \phi^{\mathsf{out}}$ *when* $\forall a \in \phi^{\mathsf{out}}$*:* $\mathsf{supp}(\gamma(a)) \subseteq \phi^{\mathsf{in}}$*.*

Definition (syntactic category $\mathcal{C}_{syn}$)

- objects are sets of names ϕ,
- morphisms are name assignations,
- composition of $\phi^{\mathsf{in}} \vdash \gamma : \phi$ and $\phi \vdash \delta : \phi^{\mathsf{out}}$: $\gamma \circ \delta$ is the partial map from $a \in \phi^{\mathsf{out}}$ to $\delta(a)\{\gamma\}$. We then have $\phi^{\mathsf{in}} \vdash \gamma \circ \delta : \phi^{\mathsf{out}}$,
- identity of $\{a_1, \ldots, a_k\}$ is the map $[a_1 \mapsto a_1] \cdots [a_k \mapsto a_k]$.

The OGS LTS ($\mathcal{L}_{\mathsf{OGS}}$)

Definition (Configurations)

$\mathbb{G}; \mathbb{H} \in \textsc{Conf}$ are either passive of the shape $\langle \gamma \rangle$, or active of the shape $\langle \mathtt{N} \mid \gamma \rangle$ with

- $\mathtt{N}$ a named term ($[c]\mathtt{T}$ with c a continuation name);
- γ a name assignation.

The OGS LTS ($\mathcal{L}_{\text{OGS}}$)

Definition (Configurations)

$\mathbb{G}; \mathbb{H} \in \textsc{Conf}$ are either passive of the shape $\langle \gamma \rangle$, or active of the shape $\langle \mathtt{N} \mid \gamma \rangle$ with

- $\mathtt{N}$ a named term ($[c]\mathtt{T}$ with c a continuation name);
- γ a name assignation.

decomp(NF) *transform normal forms into a pair* $(\mathbf{m}, \gamma)$*:*

$$\mathbf{decomp}(\mathtt{K}[x]) \triangleq \{(\overline{x}(c), [c \mapsto \mathtt{K}]) \mid c \in \mathrm{CNames}\}$$

recomp$(\mathbf{m}, \gamma)$ *apply the substitution from* γ *to get a named term:*

$$\mathbf{recomp}(x(c), \gamma) \triangleq [c]\gamma(x)$$

The OGS LTS ($\mathcal{L}_{\mathsf{OGS}}$)

Definition (Configurations)

$\mathbb{G}; \mathbb{H} \in \textsc{Conf}$ are either passive of the shape $\langle \gamma \rangle$, or active of the shape $\langle \mathtt{N} \mid \gamma \rangle$ with

- $\mathtt{N}$ a named term ($[c]\mathtt{T}$ with c a continuation name);
- γ a name assignation.

decomp(NF) *transform normal forms into a pair* $(\mathbf{m}, \gamma)$*:*

$$\mathbf{decomp}(\mathtt{K}[x]) \triangleq \{(\overline{x}(c), [c \mapsto \mathtt{K}]) \mid c \in \mathrm{CNames}\}$$

recomp$(\mathbf{m}, \gamma)$ *apply the substitution from* γ *to get a named term:*

$$\mathbf{recomp}(x(c), \gamma) \triangleq [c]\gamma(x)$$

$$\frac{\mathtt{T} \to \mathtt{U}}{\langle [c]\mathtt{T} \mid \gamma \rangle \xrightarrow{\mathsf{eval}}_{\mathsf{ogs}} \langle [c]\mathtt{U} \mid \gamma \rangle} \qquad \frac{(\mathbf{m}, \delta) \in \mathbf{decomp}(\mathtt{N})}{\langle \mathtt{N} \mid \gamma \rangle \xrightarrow{\mathbf{m}}_{\mathsf{ogs}} \langle \delta \cdot \gamma \rangle} \qquad \frac{\mathbf{recomp}(\mathbf{m}, \gamma) = \mathtt{N}}{\langle \gamma \rangle \xrightarrow{\mathbf{m}}_{\mathsf{ogs}} \langle \mathtt{N} \mid \gamma \rangle}$$

Link between Categories

We define the morphism $\fatsemi$ between $\mathcal{L}_{\mathsf{OGS}}$ and $\mathcal{G}$ as:

$$
\begin{array}{rcccl}
\langle \gamma \rangle & \fatsemi & \langle \phi^{\mathsf{out}} \mid \phi^{\mathsf{in}} \rangle^{\ominus} & \triangleq & \phi^{\mathsf{in}} \vdash \gamma : \phi^{\mathsf{out}} \\
\langle \mathtt{N} \mid \gamma \rangle & \fatsemi & \langle \phi^{\mathsf{out}} \mid \phi^{\mathsf{in}} \rangle^{\oplus} & \triangleq & \phi^{\mathsf{in}} \vdash \gamma : \phi^{\mathsf{out}} \wedge \phi^{\mathsf{in}} \vdash \mathtt{N}
\end{array}
$$

Use $\textsc{Conf}[\mathbb{P}]$ for the set of configurations $\mathbb{G}$ satisfying $\mathbb{G} \fatsemi \mathbb{P}$.

Link between Categories

We define the morphism $⨟$ between $\mathcal{L}_{\textsf{OGS}}$ and $\mathcal{G}$ as:

$$\begin{array}{rcccl} \langle \gamma \rangle & ⨟ & \langle \phi^{\textsf{out}} \mid \phi^{\textsf{in}} \rangle^{\ominus} & \triangleq & \phi^{\textsf{in}} \vdash \gamma : \phi^{\textsf{out}} \\ \langle \mathtt{N} \mid \gamma \rangle & ⨟ & \langle \phi^{\textsf{out}} \mid \phi^{\textsf{in}} \rangle^{\oplus} & \triangleq & \phi^{\textsf{in}} \vdash \gamma : \phi^{\textsf{out}} \wedge \phi^{\textsf{in}} \vdash \mathtt{N} \end{array}$$

Use $\textsc{Conf}[\mathbb{P}]$ for the set of configurations $\mathbb{G}$ satisfying $\mathbb{G} ⨟ \mathbb{P}$.

Theorem

The function mapping a name assignation to a strategy:

$$\phi^{\textsf{in}} \vdash \gamma : \phi^{\textsf{out}} \to (\mathcal{L}_{\textsf{OGS}}, ⨟, \langle \gamma \rangle)$$

induces a functor between $\mathcal{C}_{syn}$ and $\mathcal{C}_{sem}$.

Merging in $\mathcal{L}_{\mathsf{OGS}}$:

Let $\mathbb{G}_{\mathsf{P}} \in \textsc{Conf}[\langle \phi_{\mathsf{P}}^{\mathsf{in}} \mid \phi_{\mathsf{P}}^{\mathsf{out}} \rangle^{\kappa_{\mathsf{P}}}]$ and $\mathbb{G}_{\mathsf{O}} \in \textsc{Conf}[\langle \phi_{\mathsf{O}}^{\mathsf{in}} \mid \phi_{\mathsf{O}}^{\mathsf{out}} \rangle^{\kappa_{\mathsf{O}}}]$.
The *merging* $\mathbb{G}_{\mathsf{P}} \; {}_{\mathrm{H}_{\mathsf{PO}}}\curlyvee_{\mathrm{H}_{\mathsf{OP}}} \; \mathbb{G}_{\mathsf{O}}$ is

- $\langle (\!| \mathtt{N} \mid \eta |\!) \mid \gamma \rangle$ when one of $\mathbb{G}_{\mathsf{P}}, \mathbb{G}_{\mathsf{O}}$ is active and $\mathtt{N}$ the active term;
- $\langle \eta \mid \gamma \rangle$ when both $\mathbb{G}_{\mathsf{P}}, \mathbb{G}_{\mathsf{O}}$ are passive;

$\eta = \gamma_{\mathsf{P}}{}_{\restriction \mathrm{H}_{\mathsf{PO}}} \cdot \gamma_{\mathsf{O}}{}_{\restriction \mathrm{H}_{\mathsf{OP}}}$ and $\gamma = \gamma_{\mathsf{P}}{}_{\restriction \phi_{\mathsf{P}}^{\mathsf{out}} \backslash \mathrm{H}_{\mathsf{PO}}} \cdot \gamma_{\mathsf{O}}{}_{\restriction \phi_{\mathsf{O}}^{\mathsf{out}} \backslash \mathrm{H}_{\mathsf{OP}}}$.

Merging in $\mathcal{L}_{\mathsf{OGS}}$:

Let $\mathbb{G}_{\mathsf{P}} \in \textsc{Conf}[\langle \phi^{\mathsf{in}}_{\mathsf{P}} \mid \phi^{\mathsf{out}}_{\mathsf{P}} \rangle^{\kappa_{\mathsf{P}}}]$ and $\mathbb{G}_{\mathsf{O}} \in \textsc{Conf}[\langle \phi^{\mathsf{in}}_{\mathsf{O}} \mid \phi^{\mathsf{out}}_{\mathsf{O}} \rangle^{\kappa_{\mathsf{O}}}]$.
The *merging* $\mathbb{G}_{\mathsf{P}}\ {}_{\mathrm{H}_{\mathsf{PO}}}\curlyvee_{\mathrm{H}_{\mathsf{OP}}}\ \mathbb{G}_{\mathsf{O}}$ is

- $\langle (\!| \mathtt{N} \mid \eta |\!) \mid \gamma \rangle$ when one of $\mathbb{G}_{\mathsf{P}}, \mathbb{G}_{\mathsf{O}}$ is active and $\mathtt{N}$ the active term;
- $\langle \eta \mid \gamma \rangle$ when both $\mathbb{G}_{\mathsf{P}}, \mathbb{G}_{\mathsf{O}}$ are passive;

$\eta = \gamma_{\mathsf{P}\,\restriction \mathrm{H}_{\mathsf{PO}}} \cdot \gamma_{\mathsf{O}\,\restriction \mathrm{H}_{\mathsf{OP}}}$ and $\gamma = \gamma_{\mathsf{P}\,\restriction \phi^{\mathsf{out}}_{\mathsf{P}} \setminus \mathrm{H}_{\mathsf{PO}}} \cdot \gamma_{\mathsf{O}\,\restriction \phi^{\mathsf{out}}_{\mathsf{O}} \setminus \mathrm{H}_{\mathsf{OP}}}$.

MOGS LTS ($\mathcal{L}_{\mathsf{MOGS}}$):

Abstract machine:

$$\frac{\mathtt{T} \to \mathtt{U}}{(\!| [c]\mathtt{T} \mid \gamma |\!) \mapsto_{\mathsf{me\text{-}op}} (\!| [c]\mathtt{U} \mid \gamma |\!)}$$

$$\frac{\mathbb{A} \mapsto_{\mathsf{me\text{-}op}} \mathbb{B}}{\langle \mathbb{A} \mid \gamma \rangle \xrightarrow{\mathsf{eval}}_{\mathsf{mogs}} \langle \mathbb{B} \mid \gamma \rangle} \qquad \frac{\mathbb{A} \mapsto_{\mathsf{me\text{-}sy}} \mathbb{B}}{\langle \mathbb{A} \mid \gamma \rangle \xrightarrow{\mathsf{sync}}_{\mathsf{mogs}} \langle \mathbb{B} \mid \gamma \rangle}$$

$$\frac{\mathbf{recomp}(\mathbf{m}, \gamma) = \mathtt{N}}{\langle \eta \mid \gamma \rangle \xrightarrow{\mathbf{m}}_{\mathsf{mogs}} \langle (\!| \mathtt{N} \mid \eta |\!) \mid \gamma \rangle}$$

$$\frac{(\mathbf{m}, \delta) \in \mathbf{decomp}(\mathtt{N}) \quad \mathbf{recomp}(\overline{\mathbf{m}}, \gamma) = \mathtt{M}}{(\!| \mathtt{N} \mid \gamma |\!) \mapsto_{\mathsf{me\text{-}sy}} (\!| \mathtt{M} \mid \delta \cdot \gamma |\!)}$$

$$\frac{(\!| \mathtt{N} \mid \eta |\!) \text{ normal form} \quad (\mathbf{m}, \delta) \in \mathbf{decomp}(\mathtt{N})}{\langle (\!| \mathtt{N} \mid \eta |\!) \mid \gamma \rangle \xrightarrow{\mathbf{m}}_{\mathsf{mogs}} \langle \eta \mid \delta \cdot \gamma \rangle}$$

A Functor between $\mathcal{C}_{syn}$ and $\mathcal{C}_{sem}$

Theorem (Chain of bisimulations)

- *We have a bisimulation between* $(\mathcal{L}_{\mathsf{MOGS}}, \circledcirc)$ *and* $(\mathcal{L}_{\mathsf{OGS}}, \circledcirc)$.

A Functor between $\mathcal{C}_{syn}$ and $\mathcal{C}_{sem}$

Theorem (Chain of bisimulations)

- *We have a bisimulation between* $(\mathcal{L}_{\mathsf{MOGS}}, \circledcirc)$ *and* $(\mathcal{L}_{\mathsf{OGS}}, \circledcirc)$.
- *The function mapping* $\mathbb{G}_{\mathsf{P}\ \mathrm{H_{PO}}}||_{\mathrm{H_{OP}}}\ \mathbb{G}_{\mathsf{O}}$ *in* $\mathbb{G}_{\mathsf{P}\ \mathrm{H_{PO}}}\Upsilon_{\mathrm{H_{OP}}}\ \mathbb{G}_{\mathsf{O}}$ *is a bisimulation between the parallel composition of two copies of* $(\mathcal{L}_{\mathsf{OGS}}, \circledcirc)$ *and* $(\mathcal{L}_{\mathsf{MOGS}}, \circledcirc)$.

A Functor between $\mathcal{C}_{syn}$ and $\mathcal{C}_{sem}$

Theorem (Chain of bisimulations)

- *We have a bisimulation between* $(\mathcal{L}_{\mathsf{MOGS}}, \overset{\circ}{\circ})$ *and* $(\mathcal{L}_{\mathsf{OGS}}, \overset{\circ}{\circ})$.
- *The function mapping* $\mathbb{G}_{\mathsf{P}}\ {}_{\mathrm{H}_{\mathsf{PO}}}||_{\mathrm{H}_{\mathsf{OP}}}\ \mathbb{G}_{\mathsf{O}}$ *in* $\mathbb{G}_{\mathsf{P}}\ {}_{\mathrm{H}_{\mathsf{PO}}}\Upsilon_{\mathrm{H}_{\mathsf{OP}}}\ \mathbb{G}_{\mathsf{O}}$ *is a bisimulation between the parallel composition of two copies of* $(\mathcal{L}_{\mathsf{OGS}}, \overset{\circ}{\circ})$ *and* $(\mathcal{L}_{\mathsf{MOGS}}, \overset{\circ}{\circ})$.
- *We have a bisimulation between the parallel composition of two copies of* $(\mathcal{L}_{\mathsf{OGS}}, \overset{\circ}{\circ})$ *and* $(\mathcal{L}_{\mathsf{OGS}}, \overset{\circ}{\circ})$ *itself.*

A Functor between $\mathcal{C}_{syn}$ and $\mathcal{C}_{sem}$

Theorem (Chain of bisimulations)

- *We have a bisimulation between* $(\mathcal{L}_{\mathsf{MOGS}}, \circledcirc)$ *and* $(\mathcal{L}_{\mathsf{OGS}}, \circledcirc)$.
- *The function mapping* $\mathbb{G}_{\mathsf{P}}\ {}_{\mathrm{H}_{\mathsf{PO}}}||_{\mathrm{H}_{\mathsf{OP}}}\ \mathbb{G}_{\mathsf{O}}$ *in* $\mathbb{G}_{\mathsf{P}}\ {}_{\mathrm{H}_{\mathsf{PO}}}\curlyvee_{\mathrm{H}_{\mathsf{OP}}}\ \mathbb{G}_{\mathsf{O}}$ *is a bisimulation between the parallel composition of two copies of* $(\mathcal{L}_{\mathsf{OGS}}, \circledcirc)$ *and* $(\mathcal{L}_{\mathsf{MOGS}}, \circledcirc)$.
- *We have a bisimulation between the parallel composition of two copies of* $(\mathcal{L}_{\mathsf{OGS}}, \circledcirc)$ *and* $(\mathcal{L}_{\mathsf{OGS}}, \circledcirc)$ *itself.*

Theorem

The function mapping a name assignation to a strategy:

$$\phi^{\mathsf{in}} \vdash \gamma : \phi^{\mathsf{out}} \to (\mathcal{L}_{\mathsf{OGS}}, \circledcirc, \langle\gamma\rangle)$$

induces a functor between $\mathcal{C}_{syn}$ *and* $\mathcal{C}_{sem}$.

Conclusion

What have we done?

- Game LTS
- Parallel composition
- Semantic category (based on strategies)
- Syntactic category (based on name assignations)
- OGS LTS using name assignations
- a morphism between OGS LTS and game LTS
- a functor between the two categories

Conclusion

What have we done?

- Game LTS
- Parallel composition
- Semantic category (based on strategies)
- Syntactic category (based on name assignations)
- OGS LTS using name assignations
- a morphism between OGS LTS and game LTS
- a functor between the two categories

What we will do?

- A denotational model
- A call-by-need version